BIOE60009/BIOE70030 - Physiological Fluid Mechanics

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Contents

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IMPERIAL

List of important notation used in the course

As far as possible, the notation is consistent throughout the course. We use the following notation:

Symbol	Definition	Dimensions
$\mathbf{x} = (x_i)$	spatial position	L
(x, y, z)	Cartesian coordinates	L
(x_1, x_2, x_3)	Cartesian coordinates	L
(r, θ, z)	cylindrical coordinates	$(L, 1, L)$
(r, θ, ϕ)	spherical coordinates	$(L, 1, 1)$
$\nabla(\star)$	gradient operator	$1/L$
$\nabla \cdot (\star)$	divergence operator	$1/L$
$\nabla \times (\star)$	curl operator	$1/L$
$\nabla^2(\star)$	Laplacian operator	$1/L^2$
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in Cartesian coordinates	1
$\hat{\mathbf{x}}$, $\hat{\mathbf{r}}$, $\mathbf{e}_1$, $\mathbf{e}_r$, $\mathbf{e}_\theta$, *etc.*	unit vectors in other directions	1
t	time	T
$\mathbf{u} = (u_i)$	fluid velocity	L/T
u, v, w	velocity components in Cartesian coordinates	L/T
u_1, u_2, u_3	velocity components in Cartesian coordinates	L/T
u_r, u_θ, u_z	velocity components in cylindrical coordinates	L/T
u_r, u_θ, u_ϕ	velocity components in spherical coordinates	L/T
λ	bulk viscosity of a Newtonian fluid	$M/(LT)$
μ	dynamic viscosity of a Newtonian fluid	$M/(LT)$
ν	kinematic viscosity of a Newtonian fluid	L^2/T
ρ	fluid density	M/L^3
$\mathbf{I} = \delta_{ij}$	identity tensor	1
δ_{ij}	Kronecker delta	1
ε_{ijk}	Levi–Civita symbol (alternating tensor)	1
$\boldsymbol{\sigma}$, σ_{ij}	Cauchy stress tensor	$M/(LT^2)$
$\mathbf{d}$, d_{ij}	deviatoric stress tensor	$M/(LT^2)$
$\mathbf{e} = e_{ij}$	strain-rate tensor	$1/T$
$\boldsymbol{\tau}$, τ_{ij}	shear stress tensor	$M/(LT^2)$
p	pressure	$M/(LT^2)$
$\mathbf{f}$, f_i	body force per unit mass	L/T^2
$\mathbf{g}$	acceleration due to gravity	L/T^2
g	magnitude of $\mathbf{g}$	L/T^2
$\dot{\gamma}$	shear rate	$1/T$
Q	volumetric flow rate	L^3/T
R	flow resistance	$M/(L^4T)$
E	Young's modulus (linear elasticity)	$M/(LT^2)$

Although these notations are widely used in the literature, you should be aware that variations exist; for example, $\mathbf{v}$ is sometimes used for velocity, and $\mathbf{T}$ for the traction (stress) vector, *etc.*

Usually, when writing by hand, we underline first-order tensors (vectors) once, *e.g.*, $\underline{u}$, and second-order tensors twice, *e.g.*, $\underline{\underline{\sigma}}$. In the printed notes, tensors appear in **bold**, roman font. Roman font is also used for dimensionless groups, *e.g.*, Re rather than Re.

1 Tensors and the Constitutive Relationship of a Fluid

Our aim in this section is to write down equations that govern the mechanics of a fluid. Specifically, we will be looking at the pressure and viscous effects, and how to capture them mathematically.

1.1 Definition of a Tensor

The stress tensor is an important concept in fluid mechanics that will be introduced in Section 1.4. In order to understand it, we must first cover some tensor theory.

Definition of a tensor A tensor is a mathematical object whose components transform according to specific rules under a change of coordinate system.

Example of Tensors

Velocity $\mathbf{u}$, position $\mathbf{x}$, pressure p, density ρ, and any other physical property of the fluid.

Notation for tensors A tensor, $\mathbf{A}$, can be notated as a single entity, *i.e.* as $\mathbf{A}$, or in terms of its elements, $a_{ijk\ell\dots}$. We can write $\mathbf{A} = a_{ijk\ell\dots}$. Note:

- in general, a tensor can have any number of subscripts ("indices"), for example:
  - A 0th-order tensor (also called a scalar) has no subscript,

    $$\mathbf{A} = a, \quad \text{where } a \text{ is a scalar}$$

  - a 1st-order tensor (also called a vector) has one subscript,

    $$\mathbf{A} = (a_i), \quad \text{where the } a_i\text{'s are scalars}$$

  - a 2nd-order tensor (also called a matrix) has two subscripts ,

    $$\mathbf{A} = (a_{ij}), \quad \text{where the } a_{ij}\text{'s are scalars}$$

  - ...
- each subscript ranges from 1 to the number of dimensions. Therefore in three dimensions, each subscript takes the values 1, 2 or 3.

Examples of 0th-, 1st, and 2nd-order Tensors

- 0th-order tensor: fluid density (ρ), fluid dynamic viscosity(μ), fluid pressure (p)
- 1st-order tensor: (Cartesian) velocity vector

  $$\mathbf{u} = u_i = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \equiv \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \text{for } i \in \{1, 2, 3\}$$

- 2nd-order tensor: Cauchy stress tensor

  $$\boldsymbol{\sigma} = \sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \quad \text{for } i, j \in \{1, 2, 3\}$$

1.2 Two Special Tensors

There are two special tensors which are particularly useful for writing dot products and cross products in tensor notation.

- **The Kronecker delta, δ_{ij}:** This is a 2nd-order tensor representing the identity matrix, $\mathbf{I}$. It is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} . \tag{1.1}$$

Comments

1. The Kronecker delta tensor exists in any dimension, representing the identity matrix

   For a 2×2 matrix: $\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ For a 3×3 matrix: $\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. Since δ_{ij} is equivalent to the $\mathbf{I}$, the following properties hold:
   - $\delta_{ij} x_j = x_i$ (when act on a vector, the vector does not change)
   - $\delta_{ij} a_{ij} = \sum a_{ii} = \text{tr}(\mathbf{A})$ (when act on a matrix, yielding the sum of diagonal elem.)

- **The alternating tensor, ε_{ijk}:** This is also known as the Levi-Civita tensor, the permutation tensor, and the antisymmetric tensor. It is a 3rd-order tensor, and it is only defined in three dimensions (unlike the Kronecker delta). It is defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\} \\ -1 & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\} \\ 0 & \text{if } \{i,j,k\} \text{ is not a permutation of } \{1,2,3\} \end{cases} . \tag{1.2a}$$

In other words:

$$\varepsilon_{ijk} = \begin{cases} 1 & \{i,j,k\} = \{1,2,3\}, \{2,3,1\}, \{3,1,2\} \\ -1 & \{i,j,k\} = \{3,2,1\}, \{2,1,3\}, \{1,3,2\} \\ 0 & \text{otherwise} \end{cases} . \tag{1.2b}$$

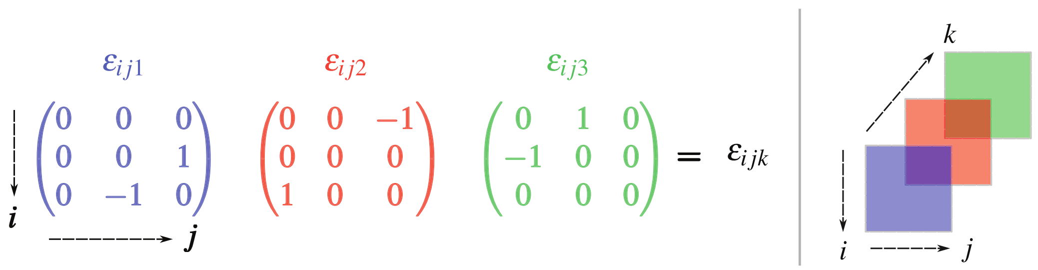


Figure 1.1: A graphical decomposition of the alternating tensor ε_{ijk} for $k \in \{1,2,3\}$.

Comments

1. Since this is a 3rd-order tensor it cannot be written very easily on a two-dimensional

sheet of paper! (see Figure 1.1, where we expanded $k = 1, 2, 3$ and stacked the tensor along the k-direction.) This is one reason why tensor notation is much more convenient when dealing with multi-dimensional quantities.

2. The alternating tensor is invariant under even permutations of its indices, and it's negative under odd permutations, that is

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = -\varepsilon_{ikj} = -\varepsilon_{jik} = -\varepsilon_{kji}. \tag{1.3}$$

3. One useful Levi-Civita and Kronecker delta (ε-δ) identity reads:

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}. \tag{1.4}$$

1.3 Tensor Operations

1.3.1 The Summation Convention

The summation convention is a convenient way of notating sums of products of terms that come up repeatedly in tensor mathematics and vector calculus. The rules of the summation convention are as follows:

1. A repeated index in a term of the expression is called a dummy index. All dummy indices are summed over (the sum is from 1 to 3 in three dimensions and from 1 to 2 in two dimensions).
2. No index can appear more than twice in a given term. Thus all dummy indices appear exactly two times in a term.
3. Indices that appear once are called free indices. The set of free indices in each term of an expression must be identical.

Example 1: Using tensor summation convention

For vectors $\mathbf{a} = (a_1, a_2, a_3)^\top$ and $\mathbf{b} = (b_1, b_2, b_3)^\top$, the dot product between $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i = a_i b_i$$

Here, $a_i b_i$ is the summation convention; $i \in \{1, 2, 3\}$ is a dummy index because it appears twice in the same term.

Example 2: Using tensor summation convention

For a tensor operation defined using the summation convention $a_{ijk} b_j c_{k\ell} + d_{i\ell}$ (where $i, j, k, \ell \in \{1, 2, 3\}$), we say i and ℓ are free indices, j and k are dummy indices.

By expanding the summation convention, and annotated as follows:

$$\overbrace{\underbrace{\left(\sum_{k=1}^{3} \underbrace{\sum_{j=1}^{3} a_{ijk} b_j}_{\text{step ①}} c_{k\ell} \right)}_{\text{step ②}} + d_{i\ell}}^{\text{step ③}} .$$

- step ①: Compute $a_{ijk} \cdot b_j$, the dummy index is j. This step eliminates the dimension j, which results in a product of the dimension $(i \times k)$.
- step ②: Compute the dot product between the result from step ① and $c_{k\ell}$. The dummy index is k. This step results in a product of the dimension $(i \times \ell)$, since $(i \times k) \cdot (k \times \ell) \to (i \times \ell)$.
- step ③: Perform summation between two tensors of the same dimension $(i \times \ell)$.

Over the operation, i and ℓ can freely range from 1 to 3, hence they are the free indices.

1.3.2 Tensor Operations in Linear Algebra

Inner Product The **inner product** of tensors is defined through *index contraction*. This operation *reduces* the order of tensors by summing over a common index.

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the inner product is defined as

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_i b_i, \tag{1.5}$$

which produces a scalar (0^{th}-order tensor).

More generally, contraction may be used to combine tensors of different orders. For example, multiplying a second-order tensor (matrix) $\mathbf{A}$ by a vector $\mathbf{x}$ yields a vector,

$$(\mathbf{A}\mathbf{x})_i = A_{ij} x_j. \tag{1.6}$$

Example

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{x} = (x_i)$. Then

$$\mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{3} a_{1j}x_j \\ \sum_{j=1}^{3} a_{2j}x_j \\ \sum_{j=1}^{3} a_{3j}x_j \end{pmatrix}$$

Using the summation convention, the above can be abbreviated as:

$$\mathbf{A}\mathbf{x} = \sum_{j=1}^{3} a_{ij} x_j = a_{ij} x_j.$$

Outer Product The **outer product**, also known as the **dyadic product**, combines two tensors without contraction and therefore *increases* the tensor order.

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, the outer product is defined as

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j. \tag{1.7}$$

The result is a second-order tensor (matrix). Unlike the dot product, no summation is implied.

Example

For vectors $\mathbf{a} = (a_1, a_2, a_3)^\top$ and $\mathbf{b} = (b_1, b_2, b_3)^\top$, the dyadic product is

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}.$$

The outer product is frequently used to construct tensors from vector bases. For example, a 2$^{\text{nd}}$-order tensor $\mathbf{A}$ may be expressed as

$$\mathbf{A} = A_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j, \tag{1.8}$$

where $\{\hat{\mathbf{e}}_i\}$ is an orthonormal basis.

Cross Product The **cross products** of tensors: involves the use of the alternating tensor

$$(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k. \tag{1.9}$$

Example

Using tensor cross product rule to prove $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \big(\varepsilon_{ijk}\, a_j\ \ (\mathbf{b} \times \mathbf{c})_k\big) &= \big(\varepsilon_{ijk}\, a_j\, \varepsilon_{k\ell m}\, b_\ell\, c_m\big) \\
&= (\ \underbrace{\varepsilon_{kij}\, \varepsilon_{k\ell m}}_{\text{since } \varepsilon_{kij}=\varepsilon_{ijk}}\ \ a_j\, b_\ell\, c_m) \\
&= [\underbrace{\big(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\big)}_{\text{by } \varepsilon\text{-}\delta \text{ identity}}\, a_j\, b_\ell\, c_m] \\
&= \big(a_j b_i c_j - a_j b_j c_i\big) \\
&= (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} \qquad \checkmark
\end{aligned}$$

1.3.3 Differential Operators in Index Notation

- The **gradient** of a scalar field $\phi(\mathbf{x})$ produces a vector:

$$(\boldsymbol{\nabla}\phi)_i = \frac{\partial \phi}{\partial x_i}. \tag{1.10}$$

  The gradient of a vector field $\mathbf{f}(\mathbf{x})$ produces a matrix ("Jacobian matrix"):

$$(\boldsymbol{\nabla}\mathbf{f})_{ij} = \frac{\partial f_i}{\partial x_j}. \tag{1.11}$$

- The **divergence** of a vector field $\mathbf{f}(\mathbf{x})$ produces a scalar:

$$\boldsymbol{\nabla} \cdot \mathbf{f} = \frac{\partial f_i}{\partial x_i}. \tag{1.12}$$

  The divergence of a 2$^{\text{nd}}$-order tensor $\mathbf{A}$ produces a vector:

$$(\boldsymbol{\nabla} \cdot \mathbf{A})_i = \frac{\partial A_{ij}}{\partial x_j}. \tag{1.13}$$

- The **curl** of a vector field $\mathbf{f}(\mathbf{x})$ can be written using the alternating tensor

$$(\nabla \times \mathbf{f})_i = \varepsilon_{ijk} \frac{\partial f_k}{\partial x_j}. \tag{1.14}$$

Comments

There are several identities in vector calculus that can be proved using these special tensors, for example

$$\begin{aligned}
\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) &= \mathbf{0}, \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - \nabla^2 \mathbf{f}, \\
\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{b}) &= (\boldsymbol{\nabla} \cdot \mathbf{b})\, \mathbf{a} + (\mathbf{b} \cdot \boldsymbol{\nabla})\, \mathbf{a} - (\boldsymbol{\nabla} \cdot \mathbf{a})\, \mathbf{b} - (\mathbf{a} \cdot \boldsymbol{\nabla})\, \mathbf{b}, \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{f}) &= 0.
\end{aligned}$$

You should be able to prove/verify these identities in tensor notation.

1.4 The Stress Tensor

Cauchy showed that the state of stress at a point in a continuum body is completely defined by a 2nd-order tensor, namely, the Cauchy stress tensor[1], $\boldsymbol{\sigma} = \sigma_{ij}$, which has 9 scalar components:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}. \tag{1.15}$$

However, the stress (force per unit area) itself is a vector (1st-order tensor) defined on a fluid element's surface. This stress vector is known as the traction, denoted by $\mathbf{t} = t_i$ and expressed as follows:

$$\begin{aligned} &\text{(in tensor notation)} \qquad t_i = \sigma_{ij} n_j, \\ &\text{(in vector notation)} \qquad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \end{aligned} \tag{1.16}$$

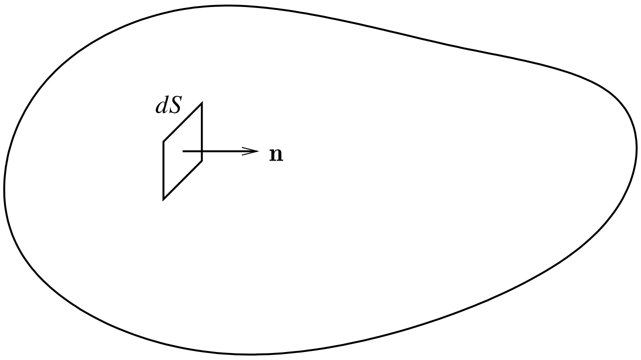
where $\mathbf{n} = n_j$ is the unit normal vector to the surface.

Thus, we can interpret the physical meaning of the stress tensor component σ_{ij} as "the i^{th} component of the traction vector acting on the surface with its unit normal vector $\mathbf{e}_j$". For example:

- σ_{12} denotes the force in the x_1-direction acting on a surface with a normal in the x_2-direction. This component represents a shear stress.
- σ_{33} denotes the force in the x_3-direction acting on a surface with a normal in the x_3-direction. This component represents a normal stress.

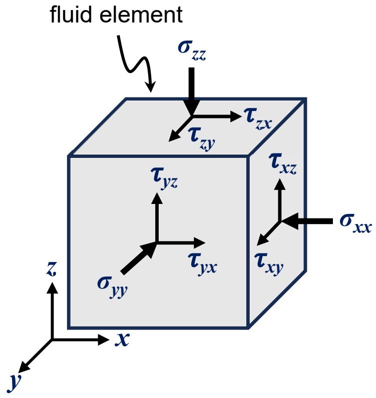
Comments

- The traction $\mathbf{t}$ is the force per unit area exerted by the fluid on the right-hand side of the small imaginary surface $\mathrm{d}S$ shown in the figure below, upon the fluid on the left-hand side of $\mathrm{d}S$.



- The stresses acting on opposite sides of a surface (*i.e.* on the surfaces with normals $\mathbf{n}$ and $-\mathbf{n}$) are equal and opposite. This is required for linear equilibrium within the fluid.
- The stress tensor is symmetric, *i.e.* $\sigma_{ij} = \sigma_{ji}$. This is required for rotational equilibrium within the fluid, and can be derived from the principle of conservation of angular momentum.
- The elements on the principal diagonal of the stress tensor matrix are called the normal stresses. The other six elements are called the shear stresses.

[1] named after Augustin-Louis Cauchy (1789-1857).



- Finally, note that these force calculations assume the components of the stress tensor are uniformly distributed over the faces of the fluid element, which is often idealised as an infinitesimal unit cube, as shown above.

Normal Stress We can decompose the stress tensor $\boldsymbol{\sigma}$ into diagonal elements and off-diagonal elements. The diagonal elements, σ_{ii}, are known as the normal stresses, and their mean defines the hydrostatic pressure:

$$p = -\frac{1}{3}\text{tr}\,(\boldsymbol{\sigma}) = -\frac{1}{3}\sigma_{ii}. \tag{1.17}$$

This equation gives us a method by which we can (at least in our imagination) think about measuring the pressure at a particular point in the fluid. We consider three small, mutually orthogonal planes passing through the point (aligned perpendicular to the x, y and z directions) and measure the three forces on the three surfaces. Dividing each force by the area of the respective plane leads to the stresses on the surfaces, which are, respectively,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}, \quad \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix}.$$

The normal components of the respective stresses are σ_{11}, σ_{22} and σ_{33}, and hence the pressure is the average of the three normal components of the stresses. The interpretation of the pressure is different for compressible and incompressible fluids:

- **Compressible fluids:** From classical thermodynamics, it is known that we can define the pressure of the fluid as a parameter of state, making use of an equation of state (for example, $p = \rho RT$ for an ideal gas).

- **Incompressible fluids:** For an incompressible fluid the pressure p is an independent, purely dynamical, variable.

Deviatoric Stress Deviatoric stress is the traceless part of the stress tensor $\boldsymbol{\sigma}$ (Note: the deviatoric stress is NOT simply the off-diagonal elements in $\boldsymbol{\sigma}$, but includes both diagonal and shear components.) The deviatoric stresses arise from the fluid motion, hence they are actually the shear stresses on each fluid particle. By Equation (1.17), we know

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}, \tag{1.18}$$

where $\mathbf{d} = d_{ij}$ is called the deviatoric stress tensor. In a fluid at rest, we have $d_{ij} = 0$, and thus $\sigma_{ij} = -p\delta_{ij}$ (hydrostatic stresses only), so in this case $\boldsymbol{\sigma}$ is a multiple of the identity matrix.

1.5 The Constitutive Relationship of Newtonian Fluid

The constitutive relationship is an equation that describes the relationship between the stress tensor and the kinematic state of the fluid. It is found from experiments and governs the mechanical behaviour of the fluid, that is the rheology of the fluid. Together with the equations of mass and momentum conservation, this closes the problem for the velocity and pressure fields. Every fluid obeying the continuum approximation has a constitutive relationship, which can be thought of as a definition of its mechanical properties.

We can now formulate a definition of a Newtonian fluid by stating its constitutive relationship. The deviatoric part of the stress tensor, $\mathbf{d}$, is a linear function of the 9 velocity gradients, $\nabla \mathbf{u} = \left(\frac{\partial u_j}{\partial x_i}\right)$, for $i, j \in \{1, 2, 3\}$. This implies

$$d_{ij} = C_{ijk\ell} \frac{\partial u_\ell}{\partial x_k},$$

for some unknown scalars in the 4$^{\text{th}}$-order tensor $C_{ijk\ell}$.

We shall now explore $C_{ijk\ell}$. There are 4 free indices in $C_{ijk\ell}$, where $i, j, k, \ell \in \{1, 2, 3\} \to$ there are $3^4 = 81$ independent coefficients! However, we can simplify these coefficients, reducing them from 81 to 2.

- The fluid is homogeneous, *i.e.*, $\boldsymbol{\sigma}$ does not depend explicitly on $\mathbf{x}$, $C_{ijk\ell}$'s are constant in space.
- By minor symmetry: $C_{ijk\ell} = C_{jik\ell}$ and $C_{ijk\ell} = C_{ij\ell k}$, which reduces the number of independent constants from 81 to 36.
- By major symmetry: $C_{ijk\ell} = C_{k\ell ij}$, which reduces the number of independent material constants from 36 to 21.
- By fluid isotropy: fluid behaves the same in any direction, $C_{ijk\ell}$ remains invariant under rotations, which reduces the number of independent material constants from 21 to 2; Namely, λ and μ. Therefore, the stress tensor

$$\text{(in tensor notation)} \qquad \sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right),$$
$$\text{(in vector notation)} \qquad \boldsymbol{\sigma} = -p\mathbf{I} + \lambda\left(\nabla \cdot \mathbf{u}\right)\mathbf{I} + 2\mu\mathbf{e}, \tag{1.19}$$

where λ is the bulk viscosity of the fluid and μ is the dynamic shear viscosity.

In Equation 1.19, $\mathbf{e} = e_{ij}$ is the strain rate tensor, given by

$$\text{(in tensor notation)} \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right),$$
$$\text{(in vector notation)} \qquad \mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right). \tag{1.20}$$

We can also write $\mathbf{e}$ as a full matrix:

- In Cartesian coordinates

$$\mathbf{e} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{pmatrix} \tag{1.21}$$

- In cylindrical polar coordinates

$$\mathbf{e} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\ \frac{1}{2}\left(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) & \frac{\partial u_z}{\partial z} \end{pmatrix}, \tag{1.22}$$

- In spherical polar coordinates

$$\mathbf{e} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & \frac{1}{2}\left(r\frac{\partial(u_\phi/r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}\right) \\ \frac{1}{2}\left(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial(u_\phi/\sin\theta)}{\partial \theta}\right) \\ \frac{1}{2}\left(r\frac{\partial(u_\phi/r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi}\right) & \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial(u_\phi/\sin\theta)}{\partial \theta}\right) & \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r + u_\theta\cot\theta}{r} \end{pmatrix}. \tag{1.23}$$

Comments

- For an incompressible fluid, the bulk viscosity λ does not contribute due to the mass conservation. In this case, the stress tensor

$$\text{(in vector notation)} \quad \boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e},$$
$$\text{(in tensor notation)} \quad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right). \tag{1.24}$$

  In Cartesian coordinates, let $u_1 = u$, $u_2 = v$, $u_3 = w$, the matrix form of $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \begin{pmatrix} -p + 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p + 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & -p + 2\mu\frac{\partial w}{\partial z} \end{pmatrix} \tag{1.25}$$

  Since we often consider incompressible Newtonian fluids, for which there is only one viscosity parameter, μ, it is common to refer to the dynamic viscosity, or just viscosity.

  Sometimes, it is more convenient to define the kinematic viscosity: $\nu = \frac{\mu}{\rho}$.

- Normal stresses: From Equation 1.25, the normal stresses are the diagonal elements in $\boldsymbol{\sigma}$:

$$\sigma_{\text{normal}} = -p + 2\mu\frac{\partial u_i}{\partial x_i}.$$

  However, if the fluid is incompressible, by the conservation of mass (continuity), $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$; Therefore, we can simplify the normal stresses to $\sigma_{\text{normal}} = -p$.

- Shear stresses: From Equation 1.25, the shear stresses are the off-diagonal elements in $\boldsymbol{\sigma}$:

$$\sigma_{\text{shear}} = \tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

  Under scenarios/assumptions *e.g.* 2-D flow, or flow is fully developed in a certain direction, we can further simplify the expression of the shear stresses.

- **Inviscid fluids:** An incompressible fluid is said to be inviscid if $\mu = 0$. The constitutive law (1.24) becomes

$$\begin{aligned} &\text{(in vector notation)} \quad && \boldsymbol{\sigma} = -p\mathbf{I}, \\ &\text{(in tensor notation)} \quad && \sigma_{ij} = -p\delta_{ij}, \end{aligned} \tag{1.26}$$

  and thus the stress in the fluid is not affected by the fluid motion. There are no truly inviscid fluids in nature, but in certain cases, it is appropriate to approximate by an inviscid fluid, for example for a fast-flowing, low-viscosity fluid.

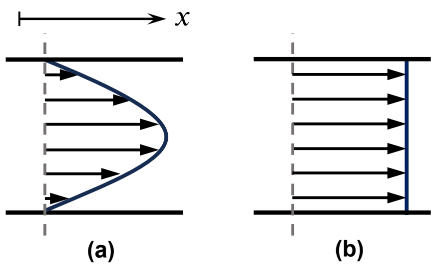


Figure 1.2: The velocity profile of flow between two parallel plates when the fluid is (a) affected by viscosity, (b) inviscid.

1.6 Non-Newtonian Fluids

Some fluids exhibit significant non-Newtonian qualities, as reflected in their constitutive relationship, which describes them. Some categories of non-Newtonian fluids:

- **Non-constant viscosity:** the viscosity varies with the shear stress τ [Pa] and shear rate $\dot{\gamma}$ [1/s]. The viscosity $\mu = \delta\tau/\delta\dot{\gamma}$.

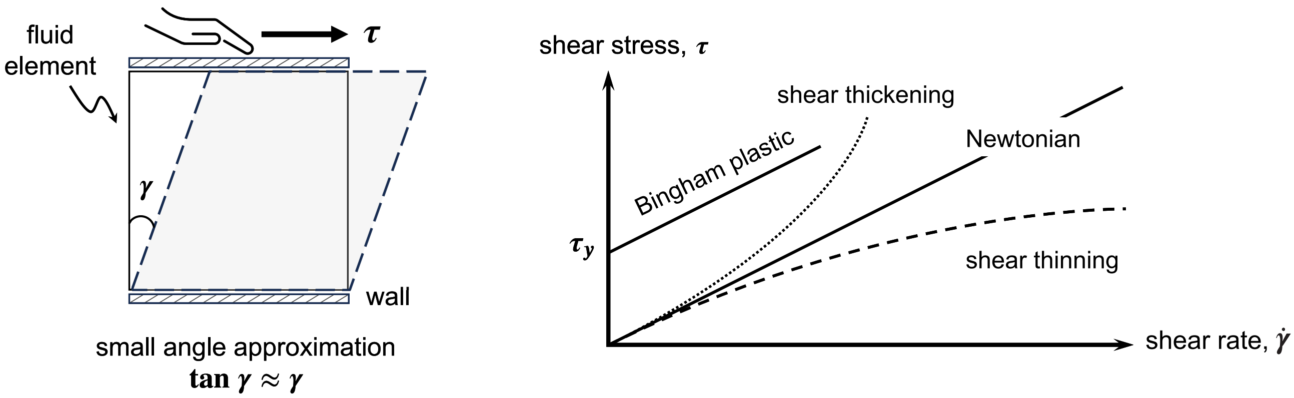


Figure 1.3: Left: the concept of shear strain γ in a simple shear flow; Right: the rheological behaviour of viscous fluids can be classified by the shear stress - shear rate ($\dot{\gamma} = \mathrm{d}\gamma/\mathrm{d}t$) relations.

  - Shear thickening: μ increases with shear rate - *e.g.*, cornstarch paste;
  - Shear thinning: μ decreases with shear rate - *e.g.*, ketchup, blood;
  - Bingham plastic: a yield stress τ_y impedes the fluid flow until $\tau > \tau_y$.

- **Viscoelastic fluids:** These fluids have a memory, and in this case, the formula for the deviatoric part of the stress tensor involves an integral over the previous states of the fluid.

- **Anisotropic fluids:** These are fluids that have different properties in different directions. A physiological example of this is blood, which contains red blood cells. When the blood is flowing the cells align, meaning the effective viscosity is much less in the direction of flow and more in the transverse direction.

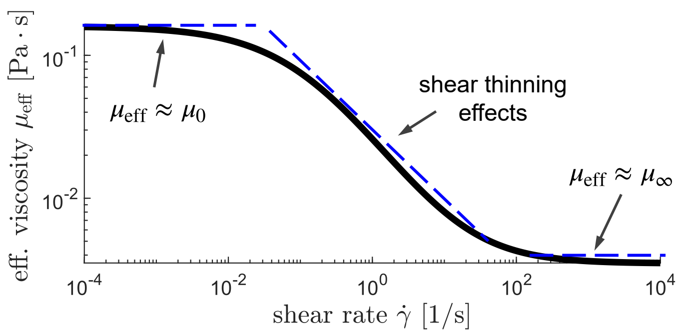
Blood Although the blood is frequently modelled as a Newtonian fluid, it exhibits shear-thinning behaviours. The non-Newtonian behaviours of blood are due to the cell suspension (rather than the plasma), hence, the viscosity is hematocrit dependent.

The blood viscosity can be modelled by the non-Newtonian Carreau-Yasuda model: the effective (apparent) viscosity μ_{eff} is expressed as a function of shear rate $\dot{\gamma}$ (see plot below):

$$\mu_{\text{eff}}(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\left[1 + (k\dot{\gamma})^a\right]^{\frac{n-1}{a}},$$

where

- μ_0 is the zero-shear viscosity,
- μ_∞ is the high-shear viscosity,
- k is the characteristic time constant,
- $\dot{\gamma}$ is the shear rate,
- n, a are constants ("power-law index" and "Yasuda exponent").



1.7 Examples

We can find the stress $\boldsymbol{\tau}$ (the force per unit area experienced by the fluid) on a surface with unit normal vector $\mathbf{n}$ using the formula (1.16), which states

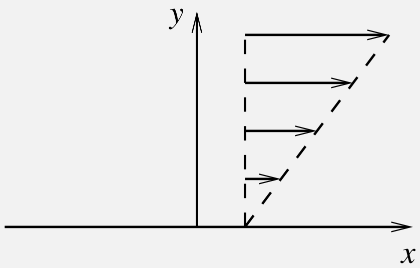
$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}.$$

Example: Shear flow

Consider an incompressible Newtonian fluid flowing in $y \geq 0$ in Cartesian coordinates (x, y, z) in a uniform pressure field $p = p_0$ with p_0 constant, and whose velocity components are respectively given by

$$u = ky, \quad v = 0, \quad w = 0,$$

(it may be shown these satisfy the Navier-Stokes and continuity equations). Find the stress tensor at a general location. Hence find the stress at a general location on an imaginary surface with unit normal vector $\mathbf{n} = (n_x, n_y)$. In this question, you may neglect gravity.



Answer: The stress tensor is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e},$$

where the rate-of-strain tensor $\mathbf{e}$ is given by

$$\mathbf{e} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix}.$$

Hence the stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e} = \begin{pmatrix} -p_0 & k\mu \\ k\mu & -p_0 \end{pmatrix}.$$

Thus for a general unit vector

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\mathbf{n} = \begin{pmatrix} -p_0 & k\mu \\ k\mu & -p_0 \end{pmatrix}\begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} -p_0 n_x + k\mu n_y \\ k\mu n_x - p_0 n_y \end{pmatrix}.$$

Comments:

- For example with $\mathbf{n} = \mathbf{j}$, we have $\mathbf{t} = (k\mu, -p_0)$. This makes sense as there is a background pressure p_0 pressing into the surface and a stress $k\mu$ arising from the shear flow and running along the surface.
- Likewise, if $\mathbf{n} = \mathbf{i}$, we have $\mathbf{t} = (-p_0, k\mu)$. This formula cannot be compared with the formula from Mechanics 2 Fluids as it would not be possible to put in a rigid surface with normal $\mathbf{n} = \mathbf{i}$ without changing the flow.

Example: Poiseuille flow

Consider Poiseuille flow in a circular cylinder with velocity components

$$u_r = 0, \quad u_\theta = 0, \quad u_z = \frac{G}{4\mu}\left(a^2 - r^2\right),$$

where $G = -\partial p/\partial z$ is the axial pressure gradient. Find the stress tensor at a general point and comment on it.

Answer: We have the strain rate tensor given by (1.22) as

$$\mathbf{e} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(r\dfrac{\partial(u_\theta/r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial r} + \dfrac{\partial u_r}{\partial z}\right) \\ \dfrac{1}{2}\left(r\dfrac{\partial(u_\theta/r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}\right) & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{u_r}{r} & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial r} + \dfrac{\partial u_r}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial\theta}\right) & \dfrac{\partial u_z}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\dfrac{Gr}{4\mu} \\ 0 & 0 & 0 \\ -\dfrac{Gr}{4\mu} & 0 & 0 \end{pmatrix},$$

and hence

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e} = \begin{pmatrix} -p & 0 & -\frac{1}{2}Gr \\ 0 & -p & 0 \\ -\frac{1}{2}Gr & 0 & -p \end{pmatrix}.$$

Incompressibility requires G to be constant, and hence $p = p_0 - Gz$, where p_0 is a constant, and giving

$$\boldsymbol{\sigma} = \begin{pmatrix} -p_0 + Gz & 0 & -\frac{1}{2}Gr \\ 0 & -p_0 + Gz & 0 \\ -\frac{1}{2}Gr & 0 & -p_0 + Gz \end{pmatrix}.$$

Hence for imaginary surfaces with normal vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{z}}$, respectively, the stress on the surface is given by

$$\begin{aligned} \mathbf{t}_r &= \boldsymbol{\sigma}\hat{\mathbf{r}} = (\mathbf{Gz} - \mathbf{p_0})\,\hat{\mathbf{r}} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Gr}\hat{\mathbf{z}}, \\ \mathbf{t}_\theta &= \boldsymbol{\sigma}\hat{\boldsymbol{\theta}} = (Gz - p_0)\,\hat{\boldsymbol{\theta}}, \\ \mathbf{t}_z &= \boldsymbol{\sigma}\hat{\mathbf{z}} = (\mathbf{Gz} - \mathbf{p_0})\,\hat{\mathbf{z}} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Gr}\hat{\mathbf{r}}. \end{aligned}$$

It is not possible to compare these formulae with the one given in Mechanics 2 Fluids, because a rigid surface with the given normal vectors could not be put in, except for the case when we find the stress at the wall ($r = a$) by setting the normal vector is $\mathbf{n} = -\hat{\mathbf{r}}$, which gives

$$\mathbf{t} = -\boldsymbol{\sigma}\hat{\mathbf{r}} = (\mathbf{p_0} - \mathbf{Gz})\,\hat{\mathbf{r}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Ga}\hat{\mathbf{z}},$$

that is a normal stress equal to minus the pressure and an axial shear stress $Ga/2$ (which is $-\mu\,\partial u_z/\partial r$, as expected).

2 The Differential Equations Governing Fluid Motion

2.1 Reynolds Transport Theorem

The continuity and the Navier-Stokes equation are derived from the Reynolds transport theorem (RTT). RTT relates the rate of change of a conserved quantity (denoted by B), in a closed system, to its rate of change within a control volume (CV) and its flux across the control surface (CS). RTT states

$$\begin{pmatrix}\text{Rate of change}\\ \text{of } B \text{ in the}\\ \text{system}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of } B \text{ in control}\\ \text{volume}\end{pmatrix} + \begin{pmatrix}\text{Net flux of } B \text{ out}\\ \text{of control volume}\\ \text{via the surface}\end{pmatrix}.$$

Mathematically,

$$\frac{\mathrm{d}B_{\text{system}}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\beta \,\mathrm{d}V + \oint_{\mathrm{CS}} \rho\beta(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}A \tag{2.1}$$

where $\beta = \mathrm{d}B/\mathrm{d}m$ is the amount of B per unit mass.

- For conservation of mass (continuity): B is the mass m, $\beta = \mathrm{d}m/\mathrm{d}m = 1$;
- For conservation of linear momentum (Navier-Stokes): B is the linear momentum $\mathbf{P} = m\mathbf{u}$, $\beta = \mathrm{d}\mathbf{P}/\mathrm{d}m = \mathbf{u}$.

Detailed derivations of the continuity and Navier-Stokes equations from RTT are provided in Appendix A.

2.2 Conservation of Mass

Using the fact that mass cannot be created or destroyed in a small control volume, and letting the size of the control volume tend to zero, we obtain

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0. \tag{2.2}$$

Incompressible fluids This is a fluid whose density $\rho(\mathbf{x}, t)$ is constant.

- Many liquids are approximately incompressible.
- The assumption of incompressibility is good for many physiological fluid flows.

For an incompressible fluid, the principle of mass conservation is equivalent to

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \tag{2.3}$$

which is the continuity equation.

2.3 Conservation of Momentum

We use Newton's second law, which states that the rate of change of momentum of a body equals the force acting on it, and consider rates of momentum change in a small control volume. The forces arising are due to the stress and any body forces acting on the fluid. Letting the size of the control volume tend to zero, we obtain Cauchy's equation,

$$\text{(in tensor notation)} \qquad \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i$$

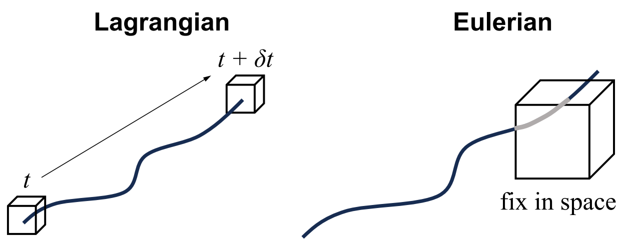
$$\text{(in vector notation)} \qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right) = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\mathbf{f} \tag{2.4}$$

where $\mathbf{f}$ is the body force per unit mass and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. The equation can be simplified by replacing the left-hand side with the material derivative,

$$\frac{D(*)}{Dt} = \frac{\partial(*)}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})(*) = \frac{\partial(*)}{\partial t} + u_j\frac{\partial(*)}{\partial x_j}. \tag{2.5}$$

Comments

- The material derivative seamlessly bridges the Lagrangian description (L.H.S. of Equation 2.5) and the Eulerian description (R.H.S. of Equation 2.5) of the fluid motion.
- Lagrangian description: keeps track of individual particles as they move through space; "go with the flow".
- Eulerian description: observe the rate of change of a property at fixed spatial locations.



Substituting Equation 2.5 into Equation 2.4

$$\text{(in tensor notation)} \qquad \rho\frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i$$

$$\text{(in vector notation)} \qquad \rho\frac{D\mathbf{u}}{Dt} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\mathbf{f}\ . \tag{2.6}$$

In each of Equations (2.4) and (2.6), the left-hand side contains terms arising from the momentum balance, whilst the right-hand side equals the force per unit volume.

Decomposing the stress into the contributions from the pressure and the deviatoric part: $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e}$ (Equation 1.24), and if the fluid is Newtonian (constant μ, hence separable from the derivatives),

$$\text{(in tensor notation)} \qquad \rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho f_i$$

$$\text{(in vector notation)} \qquad \rho\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}p + \mu\nabla^2\mathbf{u} + \rho\mathbf{f}\ , \tag{2.7}$$

which yields the celebrated Navier-Stokes (N-S)[2] equation.

For Equation 2.7, it is frequently convenient to divide both sides by the density ρ to obtain

$$\text{(in tensor notation)} \quad \frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i$$
$$\text{(in vector notation)} \quad \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \; , \tag{2.8}$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

Comments

For Equation 2.8, expanding the L.H.S. (material derivative), and we may annotate

$$\underbrace{\frac{\partial u_i}{\partial t}}_{①} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{②} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{③} + \underbrace{\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{④} + \underbrace{f_i}_{⑤}$$

① rate of change of speed (unsteady)
② convective acceleration
③ pressure gradient
④ diffusive (viscous) acceleration
⑤ body force: gravitational, EM, *etc.*

- The equation can be viewed as 3 separate equations (in 3-D), one for each spatial component.
- Term ① and ② together represent the material derivative of $\mathbf{u}$, which is the total acceleration of a fluid element.
- Term ③ and ④ represent the internal forces acting on a fluid element.
- Term ⑤ is the external force acting on a fluid element. The most common choices for $\mathbf{f}$ are $\mathbf{f} = \mathbf{g}$ (if gravity is significant) and $\mathbf{f} = \mathbf{0}$ (if gravity is unimportant).
- The N-S is non-linear due to the presence of the term ②; as a result, expansion in basis functions (*e.g.*, Fourier series) leads to coupled modal equations, and the principle of linear superposition does not apply.
- There are many possible formulations of the N-S equation. The abovementioned formulation assumes the fluid is incompressible (constant ρ) and Newtonian (constant μ, hence $\nu = \mu/\rho$ is also constant).

[2] named after Claude-Louis Navier (1785-1836) and George Gabriel Stokes (1819-1903).

2.4 The Navier-Stokes Equations In Different Coordinate Frames

What do we solve? For a three-dimensional, isothermal flow regime governed by the N-S equation, the unknown quantities to fully characterise the flow fields are the velocity components $\mathbf{u}$ in three directions, in addition to the pressure field p.

There are 4 unknowns, and from our experience, solving 4 unknowns typically requires 4 simultaneous equations. Therefore, in addition to the three Navier-Stokes equations in three orthogonal directions, we need to solve the continuity equation as the fourth equation to obtain the deterministic solution for $\mathbf{u}$ and p.

Depending on the geometry and symmetry of the problem, the governing equations may be expressed in different coordinate systems. The most commonly used systems in fluid mechanics are the Cartesian, cylindrical, and spherical coordinate systems, as illustrated in Figure 2.1. An appropriate choice of coordinate system can significantly simplify both the mathematical formulation and the analytical or numerical solution of the governing equations.

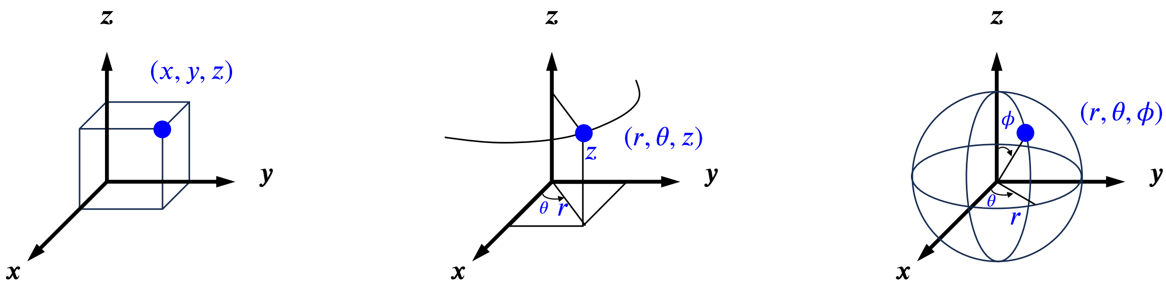


Figure 2.1: Illustration of three commonly used coordinate systems: (a) Cartesian coordinates, (b) cylindrical coordinates, and (c) spherical coordinates.

In Cartesian coordinates (x**,** y**,** z**)**

- The continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.9}$$

- The Navier-Stokes equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f_x, \tag{2.10}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho f_y, \tag{2.11}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho f_z. \tag{2.12}$$

Comments

Using tensor notation, we may write $u_i = \{u_1, u_2, u_3\}$ and $x_i = \{x_1, x_2, x_3\}$. Here, specifically within the Cartesian coordinates, we write $u_i = \{u, v, w\}$ and $x_i = \{x, y, z\}$.

i

In cylindrical coordinates (r, θ, z)

- The continuity equation

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0, \tag{2.13}$$

- The Navier-Stokes equations

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) + \rho f_r, \tag{2.14}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + \rho f_\theta, \tag{2.15}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\nabla^2 u_z + \rho f_z, \tag{2.16}$$

Comments

- The $\nabla^2(*)$ operator is known as the 'Laplacian' of the function $(*)$. In cylindrical coordinates,the Laplacian of a function $A(r, \theta, z)$ is given by

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} + \frac{\partial^2 A}{\partial z^2}. \tag{2.17}$$

- $\mathbf{f} = (f_r, f_\theta, f_z)$ represents the body forces per unit mass acting on the fluid. In the case of gravity, we set $\mathbf{f} = \rho\mathbf{g}$, where $\mathbf{g} = (0, 0, -g)$ is the acceleration due to gravity.

In spherical coordinates (r, θ, ϕ)

- The continuity equation

$$\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0, \tag{2.18}$$

- The Navier-Stokes equations

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi^2+u_\theta^2}{r}\right)$$
$$=-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\frac{2u_r}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)+\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right)+\rho f_r, \tag{2.19}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta-u_\phi^2\cot\theta}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\nabla^2 u_\theta-\frac{u_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right)+\rho f_\theta, \tag{2.20}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r u_\phi+u_\phi u_\theta\cot\theta}{r}\right)$$
$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\mu\left(\nabla^2 u_\phi-\frac{u_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}\right)+\rho f_\phi, \tag{2.21}$$

Comments

- The $\nabla^2(*)$ operator is known as the 'Laplacian' of the function $(*)$. In spherical coordinates,the Laplacian of a function $A(r,\theta,\phi)$ is given by

$$\nabla^2 A=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\phi^2}. \tag{2.22}$$

- $\mathbf{f}=(f_r,f_\theta,f_\phi)$ represents the body forces per unit mass acting on the fluid. Again in the case of gravity we set $\mathbf{f}=\rho\mathbf{g}$, where $\mathbf{g}=(-g\cos\theta,g\sin\theta,0)$.

2.5 Other Transport Phenomena

Transport of heat, mass, and linear momentum share similar mathematical frameworks.

Transport of ...	Governing Equation	"Diffusivity"	"Source"
Heat	$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \alpha \nabla^2 T + \dot{S}_T$	$\alpha = k/\rho c_v$	$\dot{S}_T = \dot{S}_v/\rho c_v$
Mass	$\frac{\partial C}{\partial t} + (\mathbf{u} \cdot \nabla)C = \mathcal{D} \nabla^2 C + \dot{S}_C$	$\mathcal{D}$	$\dot{S}_C$
Momentum (N-S)	$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \nabla^2 \mathbf{u} + \dot{S}_v$	$\nu = \mu/\rho$	$\dot{S}_v = (-\nabla p + \rho \mathbf{f})/\rho$

2.5.1 Transport of Energy

We can also use the Reynolds Transport Theorem to perform the conservation of energy in a small control volume. The energy is the sum of the internal energy, kinetic energy and the gravitational potential energy

$$e = \hat{u} + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{g} \cdot \mathbf{x},$$

where $\hat{u}$ is the internal energy per unit mass (which is often expressed as $\mathrm{d}\hat{u} \approx c_v \, \mathrm{d}T$) and $\mathbf{g}$ is the acceleration due to gravity.

In the special case that the fluid is incompressible and Newtonian and that the specific heat c_v is constant and the fluid is homogeneous and isotropic, and, upon letting the volume of the control volume tend to zero, we obtain the energy equation:

$$\rho c_v \left(\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})T \right) = k \nabla^2 T + \Phi + \dot{S}_v, \tag{2.23}$$

where

- c_v is the specific heat at constant volume, which is the rate of change of internal energy of the fluid with respect to temperature,
- T is the temperature of the fluid,
- k is the coefficient of thermal conductivity, which is the heat flux per unit area per unit temperature gradient (by Fourier's law, which is $\mathbf{q} = -k\boldsymbol{\nabla}T$ where $\mathbf{q}$ is the flux of heat per unit area - *cf.* Fick's law), and
- Φ is the rate of heating due to viscous stresses, given by

$$\Phi = 2\mu e_{ij} e_{ij} = \mu \left(2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right). \tag{2.24}$$

- $\dot{S}_v$ is the energy production per unit volume per unit time.

Comments

- In Equation 2.23, there is an extra term, Φ, representing the heating due to viscous stresses in the fluid. It is the energy loss due to the viscous forces per unit volume of fluid per unit time. Often, this term is neglected to give the usual advection-diffusion equation. In a non-Newtonian fluid, the equation is significantly more complicated.
- Divide both sides of Equation 2.23 by ρc_v, we can write $\dfrac{k}{\rho c_v} = \alpha$; Here, α is commonly called the thermal diffusivity.
- The Prandtl number, Pr, is an important dimensionless number that measures the relative importance of the viscosity to the thermal conductivity,

$$\mathrm{Pr} = \nu/\alpha,$$

  where ν is the kinematic viscosity.

2.5.2 Transport of Mass

The transport equation of mass is given by

$$\frac{\partial C}{\partial t} + (\mathbf{u}\cdot\nabla)C = \mathcal{D}\nabla^2 C + \dot{S}_C, \tag{2.25}$$

where

- C is the concentration of the substance (solute) to transport,
- $\mathcal{D}$ is the (mass) diffusivity,
- $\dot{S}_C$ is the rate of production/consumption of the substance (the sign matters: a positive sign for production "*source*", a negative sign for consumption, "*sink*").

Comments

- Under the special case where there is no advection ($\mathbf{u}$= 0) nor solute production ($\dot{S}_C$ = 0),

$$\frac{\partial C}{\partial t} = \mathcal{D}\nabla^2 C.$$

  This equation is known as Fick's second law.
- The Péclet number, Pe, is an important dimensionless number that measures the ratio of convection to diffusion,

$$\mathrm{Pe} = \frac{UL}{\mathcal{D}},$$

  where U is the velocity, L is the characteristic length.

2.6 Solving Navier-Stokes Equations Analytically

The continuity, Navier-Stokes and other transport equations are notoriously difficult to solve. In fluid mechanics, a few problems have *analytical* solutions, meaning they can be solved exactly, *e.g.*, the solutions for Poiseuille flow, Womersley flow, and flow in a square duct.

In this course, we mainly deal with the special cases of the N-S equation – which typically involves applying the assumptions to simplify the N-S equation and the continuity equation. The following summarizes the possible assumptions:

- **Steady flow:** $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial p}{\partial t} = 0$, or $\frac{\partial (*)}{\partial t} = 0$.
- **One-dimensional flow (in x-direction only):** $\frac{\partial (*)}{\partial y} = 0$ and $\frac{\partial (*)}{\partial z} = 0$ and (usually) $v = w = 0$.

  This can be generalised in any direction. For example, blood flow in arteries can be analysed using a one-dimensional approach.
- **Two-dimensional flow (in x-y plane):** $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = \frac{\partial p}{\partial z} = 0$, or $\frac{\partial (*)}{\partial z} = 0$.

  Of course, this can be generalised to two-dimensional flow in the (x, z)- or (y, z)-planes and also two-dimensional flow in the (r, θ)-directions in cylindrical polar coordinates. For example, we may do a lab experiment between closely separated parallel plates to ensure that the flow is approximately two-dimensional (Hele-Shaw cell).
- **Axisymmetric flow:** $\frac{\partial u_r}{\partial \theta} = \frac{\partial u_\theta}{\partial \theta} = \frac{\partial u_z}{\partial \theta} = \frac{\partial p}{\partial \theta} = 0$ or $\frac{\partial (*)}{\partial \theta} = 0$.

  In addition, if there is a symmetry in a plane that includes the axis, we would also have $u_\theta = 0$. This can be generalised to axisymmetric flow in spherical coordinates (using $\frac{\partial}{\partial \phi} = 0$).
- **Spherically symmetric flow:** $\frac{\partial (*)}{\partial \theta} = 0$ and $\frac{\partial (*)}{\partial \phi} = 0$ and $u_\theta = u_\phi = 0$.
- **Fully developed flow:** $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial z} = 0$, but $\frac{\partial p}{\partial z} \neq 0$ in general.
- **Independence of a coordinate:** For example, $\partial / \partial x = 0$ if nothing in the problem depends on x and there is no reason to assume that anything should depend on x. We used this for the Stokes boundary layer problem in Section 2.7.
- **Periodic and sinusoidal flow (with period T):**

$$
\begin{aligned}
u(x, y, z, t) &= u_0(x, y, z)e^{i\omega t} + \overline{u_0(x, y, z)}e^{-i\omega t}, \\
v(x, y, z, t) &= v_0(x, y, z)e^{i\omega t} + \overline{v_0(x, y, z)}e^{-i\omega t}, \\
w(x, y, z, t) &= w_0(x, y, z)e^{i\omega t} + \overline{w_0(x, y, z)}e^{-i\omega t}, \\
p(x, y, z, t) &= p_0(x, y, z)e^{i\omega t} + \overline{p_0(x, y, z)}e^{-i\omega t},
\end{aligned}
$$

  where the angular frequency $\omega = 2\pi/T$. This assumption was made in the Stokes boundary layer example covered in Section 2.7.

The trick is to look carefully at the problem and think which (if any) of the above assumptions might be reasonable. We can try making them, and then check that the problem is still consistent. If it is consistent that is fine, but if not, it means that the assumption was wrong.

2.7 Examples

Example 1: Hagen-Poiseuille Solution

The Hagen-Poiseuille[a] solution states that the pressure drop Δp of an incompressible, Newtonian fluid flowing in a laminar regime through a long cylindrical pipe is given by

$$\Delta p = \frac{8\mu L}{\pi R^4} \cdot Q, \tag{2.26}$$

where L is the pipe length, R is the pipe radius, and Q is the volumetric flow rate (of the SI unit $[\mathrm{m}^3/\mathrm{s}]$).

Question: Derive Equation 2.26 by analytically solving the Navier-Stokes equations.

Answer: We shall first list out the essential assumptions used to simplify the equation. Namely

1. Steady flow: $\frac{\partial(\star)}{\partial t} = 0$;
2. Radial and circumferential components of the fluid velocity are zero: $u_r = u_\theta = 0$;
3. The flow is assumed to be axisymmetric $\frac{\partial(\star)}{\partial \theta} = 0$;
4. The flow is fully developed along the z-direction: $\frac{\partial \mathbf{u}}{\partial z} = 0$;
5. Negligible body force: $\mathbf{f} = 0$.

Apply these assumptions to simplify the governing equations:

- The continuity equation:

$$\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

- The r-momentum equation:

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] + \rho f_r$$

- The θ-momentum equation:

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] + \rho f_\theta$$

- The z-momentum equation:

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho f_z$$

With these assumptions, the continuity equation and θ-momentum equations are trivially satisfied (*i.e.*, LHS=RHS=0). Further, for r-momentum equation, $-\dfrac{\partial p}{\partial r} = 0$ simply implies that the pressure p is constant along the r-direction. Only the terms left in the z-momentum equation need to be solved:

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] \quad \Rightarrow \quad \frac{r}{\mu}\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

$$\xRightarrow{\int(\star)\mathrm{d}r} \quad \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + c_1 = r\frac{\partial u_z}{\partial r}$$

$$\xRightarrow{\int(\star)\mathrm{d}r} \quad \boxed{u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + c_1 \ln r + c_2},$$

where c_1 and c_2 are two constants subject to the boundary conditions:

- No-slip boundary condition: $u_z = 0$ at $r = R$; and
- Finite velocity at the centerline of the pipe (*i.e.*, a flat velocity profile): $\dfrac{\partial u_z}{\partial r} = 0$ at $r = 0$.

which yields $c_1 = 0$ and $c_2 = -\dfrac{R^2}{4\mu}$, hence,

$$u_z = \frac{1}{4\mu}(r^2 - R^2)\frac{\partial p}{\partial z}.$$

Assuming the pressure p decreases linearly from the inlet $z = 0$ to the outlet $z = L$, we may *linearize* the pressure gradient $\dfrac{\partial p}{\partial z} = -\dfrac{\Delta p}{L}$:

$$u_z = -\frac{1}{4\mu}(r^2 - R^2)\frac{\Delta p}{L}.$$

Now we can calculate the volumetric flow rate Q through the cross-section of the cylinder as

$$Q = \int u_z \,\mathrm{d}A = \int_0^R u_z \; 2\pi r \,\mathrm{d}r = \frac{\pi R^4 \Delta p}{8\mu L},$$

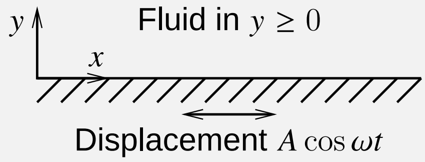
which matches the Hagen-Poiseuille solution as shown in Equation 2.26.

■

[a] named after Gotthilf Heinrich Ludwig Hagen (1797-1884) and Jean Léonard Marie Poiseuille (1797-1869).

Example 2: Stokes Boundary Layer

In this question, you will investigate the flow field generated next to a flat oscillating plate. Assume that the plate is at $y = 0$ with an incompressible Newtonian fluid of density ρ and kinematic viscosity ν in $y > 0$, and that the plate oscillates purely in the x-direction with displacement $A\cos\omega t\,\mathbf{i}$ at time t, as shown in the diagram below.



You may also assume that the plate and volume of fluid are large enough that you can neglect their boundaries (that is, the plate occupies the whole plane $y = 0$ and the fluid occupies the whole of $y > 0$) and that the plate has been oscillating long enough so that the whole fluid is oscillating periodically.

Question: Write down the equations and boundary conditions governing the flow.

Answer: The fluid is governed by the Navier-Stokes and continuity equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\nabla^2\mathbf{u}, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

At $y = 0$ the boundary conditions are

$$u = -A\omega\sin\omega t, \quad v = w = 0.$$

Question: Write down assumptions about the components of the fluid flow and their dependence on the spatial coordinates x, y and z. Use this to simplify the governing equations and boundary conditions.

Answer: We assume this is a two-dimensional flow, with $w = 0$ and all components not depending on z. In addition, we assume the velocity components do not depend on x (because translating the problem in the x-direction is the same problem). Thus we have $u(y,t)$, $v(y,t)$, $w = 0$ and $p(y,t)$.

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The first and third terms are zero by assumption, and therefore

$$\frac{\partial v}{\partial y} = 0.$$

Hence v is constant, and the boundary condition $v = 0$ at $y = 0$ implies that $v = 0$ everywhere. The three components of the Navier–Stokes equation simplify to

$$\frac{\partial u}{\partial t} + 0 + 0 + 0 = 0 + \nu\left(0 + \frac{\partial^2 u}{\partial y^2} + 0\right),$$

$$0 + 0 + 0 + 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\,(0 + 0 + 0)\,,$$

$$0 + 0 + 0 + 0 = 0 + \nu\,(0 + 0 + 0)\,.$$

Thus p is constant in space and the only non-trivial equation is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2},$$

together with boundary conditions $u = -A\omega \sin \omega t$ at $y = 0$.

Question: Assume that the flow and pressure are sinusoidal, that is $\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x})e^{\mathrm{i}\omega t} + \overline{\mathbf{U}(\mathbf{x})}e^{-\mathrm{i}\omega t}$ and $p(\mathbf{x}, t) = P(\mathbf{x})e^{\mathrm{i}\omega t} + \overline{P(\mathbf{x})}e^{-\mathrm{i}\omega t}$. Use this to simplify and solve the governing equations to find the velocity field.

Answer: The periodic assumption simplifies to $u = U(y)e^{\mathrm{i}\omega t} + \overline{U(y)}e^{-\mathrm{i}\omega t}$ (the other components are not needed). Substituting into the governing equation,

$$\mathrm{i}\omega U e^{\mathrm{i}\omega t} - \mathrm{i}\omega\overline{U(y)}e^{-\mathrm{i}\omega t} = \nu\left(\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}e^{\mathrm{i}\omega t} + \frac{\mathrm{d}^2\overline{U}}{\mathrm{d}y^2}e^{-\mathrm{i}\omega t}\right).$$

The coefficients of $e^{\mathrm{i}\omega t}$ must balance and the coefficients of $e^{-\mathrm{i}\omega t}$ must balance. Therefore

$$\mathrm{i}\omega U = \nu\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}, \quad \text{and} \quad -\mathrm{i}\omega\overline{U} = \nu\frac{\mathrm{d}^2\overline{U}}{\mathrm{d}y^2}.$$

These two equations are the complex conjugates of one another, so we only need to solve one of them, which gives the general solution

$$U = C_1 e^{\sqrt{\frac{\mathrm{i}\omega}{\nu}}y} + C_2 e^{-\sqrt{\frac{\mathrm{i}\omega}{\nu}}y}.$$

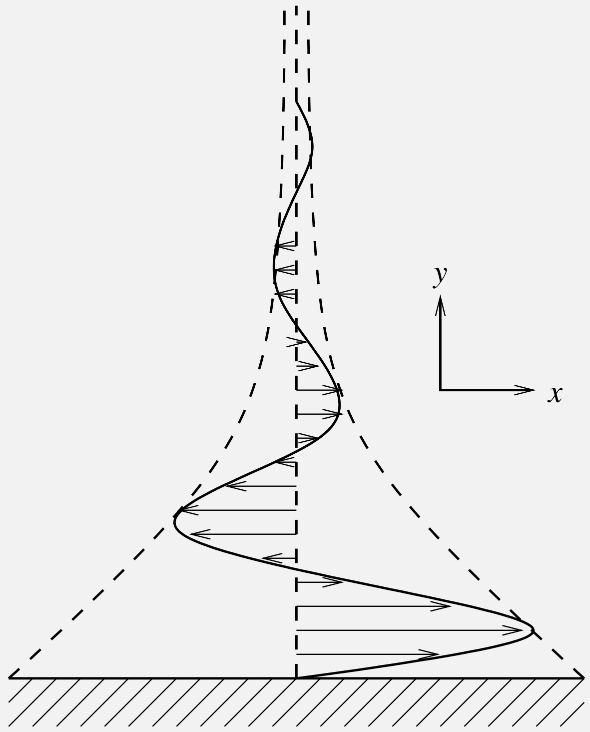
The function $e^{\sqrt{\frac{\mathrm{i}\omega}{\nu}}y} \to \infty$ as $y \to \infty$, and thus $C_1 = 0$. The boundary condition $u = -A\omega\sin\omega t = (A\omega\mathrm{i}/2)e^{\mathrm{i}\omega t} - (A\omega\mathrm{i}/2)e^{-\mathrm{i}\omega t}$ at $y = 0$. Hence $U = A\omega\mathrm{i}/2$, meaning that $C_2 = A\omega\mathrm{i}/2$. Hence

$$\begin{aligned}
u &= \frac{A\omega\mathrm{i}}{2}e^{-\sqrt{\frac{\mathrm{i}\omega}{\nu}}y+\mathrm{i}\omega t} - \frac{A\omega\mathrm{i}}{2}e^{-\sqrt{\frac{-\mathrm{i}\omega}{\nu}}y-\mathrm{i}\omega t} \\
&= -\frac{A\omega}{2\mathrm{i}}\left(e^{-(1+\mathrm{i})\sqrt{\frac{\omega}{2\nu}}y+\mathrm{i}\omega t} - e^{-(1-\mathrm{i})\sqrt{\frac{\omega}{2\nu}}y-\mathrm{i}\omega t}\right) \\
&= -A\omega e^{-\sqrt{\frac{\omega}{2\nu}}y}\frac{\left(e^{\mathrm{i}(-\sqrt{\frac{\omega}{2\nu}}y+\omega t)} - e^{-\mathrm{i}(-\sqrt{\frac{\omega}{2\nu}}y+\omega t)}\right)}{2\mathrm{i}} \\
&= -A\omega e^{-\sqrt{\frac{\omega}{2\nu}}y}\sin\left(\omega t - \sqrt{\frac{\omega}{2\nu}}y\right).
\end{aligned}$$

■

Question: Sketch the velocity profile at time t.

Answer: The velocity profile is a sinusoidal function whose amplitude decays exponentially with the distance from the surface.



This is a famous exact solution of the Navier-Stokes equations, and the flow we have found here is characteristic of the flow that exists near an oscillating boundary. Note that the flow decays away from the wall, and for this reason, it is often described as a boundary layer, and the solution is called *Stokes boundary layer* flow.

3 Analytical Solutions of the Navier-Stokes Equations

In addition to the two examples provided in Section 2.7, in this section, we shall explore two analytical solutions of the Navier-Stokes equations obtained under the physically and physiologically meaningful assumptions and boundary conditions.

- In Section 3.1, we consider steady, fully developed laminar **flow in a rectangular channel**, for which the velocity field can be derived following the principle of linear superposition.
- In Section 3.2, we examine the unsteady, pressure-driven flow in a rigid circular pipe, known as the **Womersley solution**, which extends the classical Poiseuille flow to pulsatile conditions commonly encountered in physiological flows.

3.1 Flow in a Rectangular Channel

Consider the flow in a rectangular duct (length L, width w, height h) in the Cartesian coordinate system (Figure 3.1). We want to obtain an analytical solution to the flow profile within the rectangular channel.

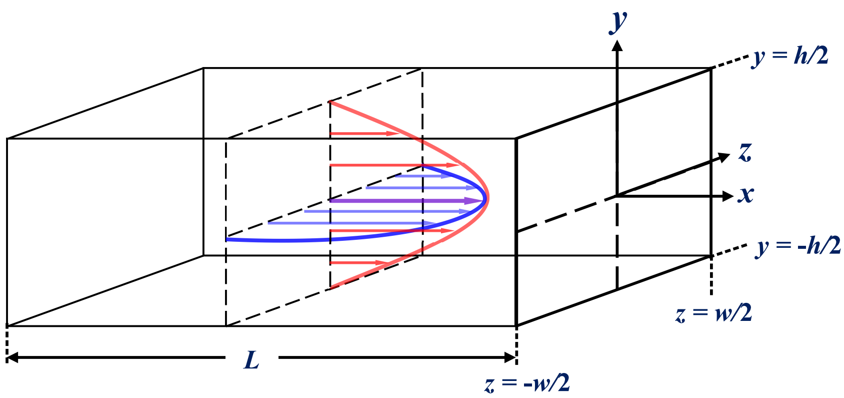


Figure 3.1: The schematic for the flow in a rectangular duct.

3.1.1 Problem Definition

Assumptions

- Fluid is homogeneous, incompressible and Newtonian with viscosity μ and density ρ;
- Flow has reached the steady state: $\partial \mathbf{u}/\partial t = 0$;
- Flow is fully developed along the x-direction: $\partial \mathbf{u}/\partial x = 0$;
- Zero velocity along the y- and z-directions: $v = 0$, $w = 0$;
- Negligible body force: $\mathbf{f} = 0$.

Boundary Conditions

- Symmetrical flow profile at $y = 0$ and $z = 0$;
- no-slip condition at the wall $y = \pm h/2$, $z = \pm w/2$.

3.1.2 Solution Procedure

Step 1 Starting from the x-component of the N-S equation, apply the assumptions:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+\cancel{u\frac{\partial u}{\partial x}}+\cancel{v\frac{\partial u}{\partial y}}+\cancel{w\frac{\partial u}{\partial z}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\cancel{\rho f_x}$$

$$\Rightarrow \quad 0=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{3.1}$$

Similarly, we can apply the assumptions to the y- and z-components of the N-S equations, all terms are cancelled out, leaving $0 = 0$ on both sides.

Step 2 Equation 3.1 is *nonhomogenous*! To solve this equation, we further assume that the solution is a combination of simple parallel plate Poiseuille flow plus some perturbation that is dependent on the walls and finite width:

$$u(y,z)=u_{\text{parabolic}}(y)+\underbrace{\phi(y,z)}_{\text{perturbation}}$$

and

$$0=\underbrace{-\frac{\partial p}{\partial x}+\mu\frac{\partial u^2_{\text{parabolic}}}{\partial y^2}}_{=0,\ \text{Poiseuille solution}}+\mu\left(\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}\right)$$

Since the first term is exactly the Poiseuille solution, the second term involves the derivatives of the perturbation function must be 0; Therefore, we now only need to find the solution of the perturbation function $\phi(y,z)$ to solve $u(y,z)$.

$$\mu\left(\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}\right)=0 \tag{3.2}$$

Step 3 Equation 3.2 is homogeneous! We can then employ the separation of variable method, for which ϕ is the product of a y-dependent function and a z-dependent function.

$$\phi(y,z)=Y(y)Z(z)$$

Therefore,

$$\begin{aligned}0&=Z(z)\frac{\partial^2 Y(y)}{\partial y^2}+Y(y)\frac{\partial^2 Z(z)}{\partial z^2}\\&=\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}\end{aligned}$$

Since each term is independent of the other term, therefore, we conclude that each term must be a constant.

$$\lambda^2=-\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}=\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}.$$

So far, We have decomposed the partial differential equation into two ordinary differential equations (ODE). We will now address each of the ODE separately. The general solutions of $Y(y)$ and $Z(z)$ are

$$\begin{aligned}Y(y)&=A_1\sin(\lambda y)+A_2\cos(\lambda y)\\Z(z)&=B_1\sinh(\lambda y)+B_2\cosh(\lambda y)\end{aligned}$$

where A_1, A_2, B_1, B_2 are constants subject to the specific boundary conditions.

Step 4 Solving $Y(y)$:

- Applying the first y-related boundary condition: the flow profile is symmetrical at $y = 0$, this implies that on the 'tip' of the velocity profile (maximum velocity), $\mathrm{d}Y/\mathrm{d}y = 0$, yielding $A_1 = 0$.
- Applying the second y-related boundary condition: at $y = \pm h/2$, $u = 0$:

$$\left.\frac{\mathrm{d}Y}{\mathrm{d}y}\right|_{y=h/2} = -A_2\lambda\ \sin\left(\lambda\frac{h}{2}\right) = 0,$$

since A_2 cannot be 0 (otherwise, the solution $Y(y)$ is nontrivial!), we know $\sin\left(\lambda\frac{h}{2}\right) = 0$, implying $\lambda\frac{h}{2} = \frac{(2n+1)\pi}{2}$ $(= \lambda_n)$, where n is a positive integer.

Step 5 Solving $Z(z)$:

- Applying the first z-related boundary condition: the flow profile is symmetrical at $z = 0$, this implies that on the 'tip' of the velocity profile (maximum velocity), $\mathrm{d}Z/\mathrm{d}z = 0$, yielding $B_1 = 0$.

Step 6 So far, the solution for $\phi(y, z)$ is

$$\begin{aligned}\phi(y,z) = Y(y)Z(z) &= \sum_{n=1}^{\infty} A_1\cos(\lambda y)\ B_2\cosh(\lambda z)\\ &= \sum_{n=1}^{\infty} A_n\cos(\lambda y)\ \cosh(\lambda z).\end{aligned} \tag{3.3}$$

where A_n is a constant that combines A_2 and B_n; however, what is the expression of A_n? We apply the final boundary: at $z = \pm w/2$, $u = 0$, this expression is equivalent to $\phi(y, \pm w/2) = u_{\text{parabolic}}$:

$$\phi(y, w/2) = \sum_{n=1}^{\infty} A_n\cos(\lambda y)\ B_2\cosh\left(\lambda\frac{w}{2}\right)\ =\ u_{\text{parabolic}}\ = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[y^2 - \left(\frac{h}{2}\right)^2\right] \tag{3.4}$$

Looking at Equation 3.4, both sides only involve one variable y - hence we are (somehow...) reassured that A_n can be found by integrating both sides of equation w.r.t. y from $-h$ to h.

Step 7 One more step to take before integration: we need to multiply both sides by a $\cos\left(\frac{\lambda_m y}{h/2}\right)$ term - this step is essential to make both sides of the equation *appropriately* periodic.

For simplicity, we write $\lambda = \frac{\lambda_n}{h/2}$ from step 4 (where $\lambda_n = \frac{(2n+1)\pi}{2}$):

$$\sum_{n=1}^{\infty}\int_{-h}^{h}\cos\left(\frac{\lambda_m y}{h/2}\right)A_n\cos(\lambda_n y)\ B_2\cosh\left(\lambda_n\frac{w}{2}\right)\ \mathrm{d}y = \int_{-h}^{h}\cos\left(\frac{\lambda_m y}{h/2}\right)\left[\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - \left(\frac{h}{2}\right)^2\right)\right]\mathrm{d}y\ .$$

On the LHS, due to the orthogonality of Fourier terms (cosine terms), all terms where $m \neq n$ will become 0! Only the term with $n = m$ will remain, this will allow us to find an expression for coefficient A_m[3].

[3] Yes, $A_m = A_n$ for $m = n$, but in our final expression the $\frac{1}{2\mu}\frac{\partial p}{\partial x}$ are separated and grouped with other terms, leaving an A_n as defined as shown below, hence, we use the subscript m to denote the solution from integration, to distinguish from A_n presented in our final solution.

Step 8 This fancy expression finally yields a result of A_m

$$A_m = \frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{h^2(-1)^n}{\lambda_n^3 \cosh\left(\frac{\lambda_n w}{h}\right)}.$$

which concludes our solution procedure. Substituting back to the expression of u, our final solution

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[y^2 - \left(\frac{h}{2}\right)^2 - \sum_{n=0}^{\infty} A_n \cos\left(\frac{\lambda_n y}{h/2}\right)\cosh\left(\frac{\lambda_n z}{h/2}\right)\right],$$

where $A_m = \dfrac{h^2(-1)^n}{\lambda_n^3 \cosh \dfrac{\lambda_n w}{h}}$, with $\lambda_n = \dfrac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}_{\geq 0}$.

3.1.3 Result and Extended Quantities

- The analytical solution of u:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[y^2 - \left(\frac{h}{2}\right)^2 - \sum_{n=0}^{\infty} A_n \cos\left(\frac{\lambda_n y}{h/2}\right)\cosh\left(\frac{\lambda_n z}{h/2}\right)\right], \quad \text{where } A_n = \frac{h^2(-1)^n}{\lambda_n^3 \cosh \frac{\lambda_n w}{h}}, \quad \lambda_n = \frac{(2n+1)\pi}{2}.$$

- The flow rate Q is found by integrating u over the area,

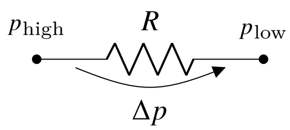
$$Q = \frac{\partial p}{\partial x}\frac{wh^3}{12\mu}\left[6\left(\frac{h}{w}\right)\sum_{n=0}^{\infty}\lambda_n^{-5}\tanh\left(\frac{\lambda_n w}{h}\right) - 1\right].$$

  Still, one needs to take 4~5 n-terms to obtain a sufficiently accurate solution of Q; A good numerical approximation of Q (10% error for $h/w \geq 0.7$):

$$Q \approx \frac{\partial p}{\partial x}\frac{wh^3}{12\mu}\left[1 - 0.6274\left(\frac{h}{w}\right)\right].$$

- The flow resistance R is found by $Q = \Delta p / R$:

$$R = \frac{\Delta p}{Q} = \frac{12\mu L}{wh^3\left[1 - 0.6274\left(\frac{h}{w}\right)\right]}.$$



  For more complex flow systems with multiple channels connected in series or in parallel (Figure 3.2), the total flow resistance R_{total} can be calculated as follows:

  - in series: $R_{\mathrm{total}} = R_1 + R_2 + R_3 + \ldots,$
  - in parallel: $\dfrac{1}{R_{\mathrm{total}}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots.$

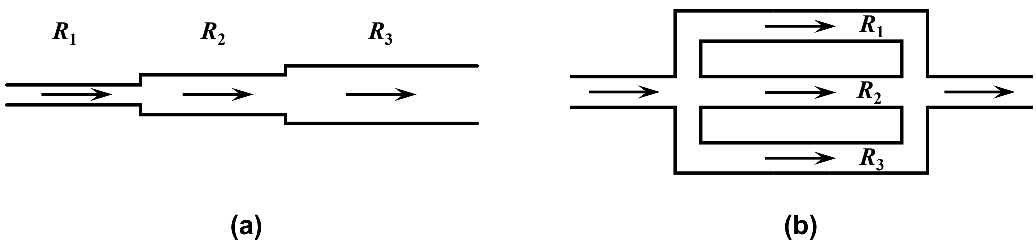


Figure 3.2: Examples of multi-channel systems: (a) three channels connected in series; (b) three channels connected in parallel.

3.2 The Womersley Flow

We consider a long, circular cylindrical channel of radius a, and work in cylindrical polar coordinates centred on the axis of the pipe. We want to obtain an unsteady analytical solution to the flow profile within the channel. The Wormsley flow[4] occurs in pipes in which there is an imposed oscillating pressure gradient; it is a good approximation to the pulsatile flow in the human cardiovascular system.

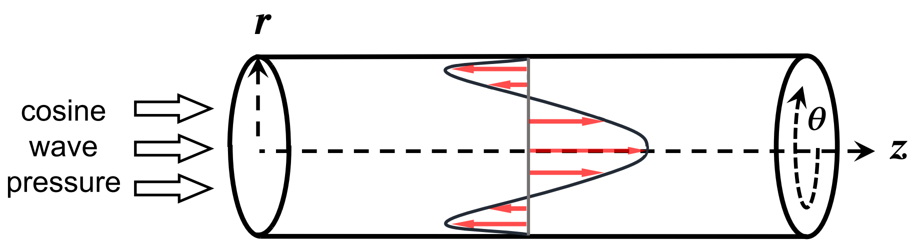


Figure 3.3: The schematic of the Womersley flow in a pipe.

3.2.1 Problem Definition

Assumptions

- Fluid is homogeneous, incompressible and Newtonian with viscosity μ and density ρ;
- Flow in a long straight tube, with a perfect circular cross-section at radius a;
- Axisymmetric about the z-axis: $\partial/\partial\theta = 0$;
- The flow is fully developed along the z-axis: $\partial \mathbf{u}/\partial z = 0$;
- No swirls: $u_\theta = 0$;
- No velocity along the radial direction: $u_r = 0$;
- Negligible body force: $\mathbf{f} = 0$.

Boundary Conditions No-slip condition on the wall, flow symmetry about the centreline. The flow is driven by a time-periodic axial pressure gradient.

3.2.2 Solution Procedure

Step 1 Starting from the z-component of the N-S equation, apply the assumptions:

$$\rho\left(\frac{\partial u_z}{\partial t} + \cancel{u_r\frac{\partial u_z}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}} + \cancel{u_z\frac{\partial u_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 u_z}{\partial z^2}}\right] + \rho \cancel{f_z}$$

$$\Rightarrow \quad \rho\frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right), \tag{3.5}$$

which is the governing equation for the Womersley flow.

[4] named after John R. Womersley (1907–1958).

Step 2 For the Womersely flow, we hypothesise that the pressure gradient $\dfrac{\partial p}{\partial z}$ is sinusoidal:

$$\frac{\partial p}{\partial z} = G_0 \cos(\omega t) = \frac{G_0}{2} e^{\mathrm{i}\omega t}, \tag{3.6}$$

where $G_0 \in \mathbb{R}^+$ is a constant, $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

Why $\dfrac{\partial p}{\partial z}$ *is a function of time?* Consider what happens to the three terms in Equation (3.5). Easy to see

- u_z is a function of t and r; p is a function of z.
- However, to maintain the equality of both sides, $\dfrac{\partial p}{\partial z}$ cannot be a function a of z, thus $p = A(t)z + B(t)$, where A and B are unknown functions of time.
- For Womersley flow, we deliberately select $A(t) = -G_0 \cos(\omega t)$, yielding Equation (3.6). Note the minus symbol here, since a negative pressure gradient is required to drive the flow from the inlet to the outlet.

Step 3 We select a trial solution of u_z of the form:

$$u_z(r,t) = U(r)e^{\mathrm{i}\omega t}, \tag{3.7}$$

where $U(t)$ is a complex-valued function (*i.e.* $U \leftrightarrow \bar{U}$). Substituting this into Equation (3.5), we obtain

$$\underbrace{\rho \mathrm{i}\omega U e^{\mathrm{i}\omega t} - \rho \mathrm{i}\omega \overline{U} e^{-\mathrm{i}\omega t}}_{\rho \frac{\partial u_z}{\partial t}} = \underbrace{\frac{G_0}{2} e^{\mathrm{i}\omega t} + \frac{G_0}{2} e^{-\mathrm{i}\omega t}}_{\frac{\partial p}{\partial z}} + \underbrace{\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial U}{\partial r}\right) e^{\mathrm{i}\omega t} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \overline{U}}{\partial r}\right) e^{-\mathrm{i}\omega t}}_{\mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right)}$$

$$\Rightarrow \quad \left\{\rho \mathrm{i}\omega U - \frac{G_0}{2} - \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)\right\} e^{\mathrm{i}\omega t} + \left\{-\rho \mathrm{i}\omega \overline{U} - \frac{G_0}{2} - \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \overline{U}}{\partial r}\right)\right\} e^{-\mathrm{i}\omega t} = 0$$

$$\Rightarrow \quad \mathrm{i}\omega\rho U - \frac{G_0}{2} - \mu \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}U}{\mathrm{d}r}\right) = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}^2 U}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}r} - \frac{\mathrm{i}\omega\rho}{\mu} U = -\frac{G_0}{2\mu}. \tag{3.8}$$

Equation (3.8) is a linear, second-order, non-constant-coefficient, non-homogeneous ordinary differential equation for U.

Step 4 We can therefore solve Equation (3.8) as a complementary function U_{cf} plus a particular integral U_{pi}. The particular integral is given by

$$U_{\mathrm{pi}} = \frac{G_0}{2\mathrm{i}\omega\rho}. \tag{3.9}$$

To find the complementary function, we must solve

$$\frac{\mathrm{d}^2 U_{\mathrm{cf}}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}U_{\mathrm{cf}}}{\mathrm{d}r} - \frac{\mathrm{i}\omega\rho}{\mu} U_{\mathrm{cf}} = 0, \tag{3.10}$$

and substituting $s = (\sqrt{-\mathrm{i}\omega/\nu})r$ and simplifying, we remove the coefficients on the left-hand side to obtain

$$\frac{\mathrm{d}^2 U_{\mathrm{cf}}}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}U_{\mathrm{cf}}}{\mathrm{d}s} + U_{\mathrm{cf}} = 0. \tag{3.11}$$

This equation is the standard form of a special equation called Bessel equation.

Comments: There are two linearly independent solutions for the Bessel equation, namely

- Bessel function of the first kind: $J_n(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}$.
- Bessel function of the second kind: $Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$.

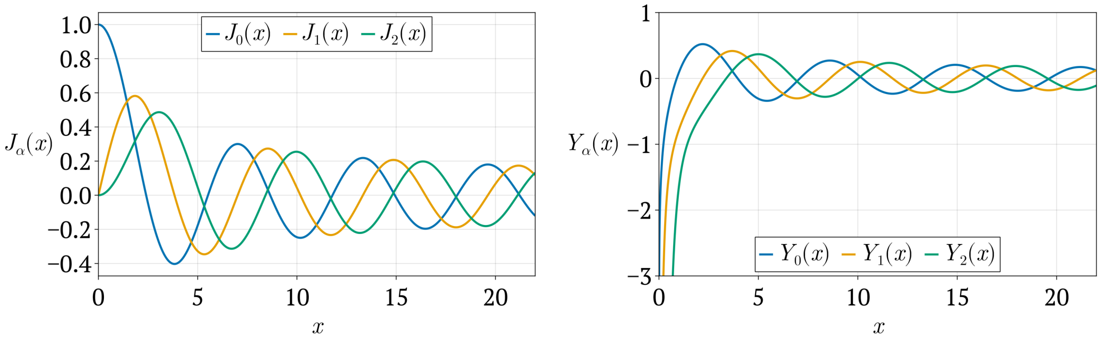


Figure 3.4: (a) Bessel functions of the first kind, J_0, J_1 and J_2; (b) Bessel functions of the second kind, Y_0, Y_1 and Y_2 (Wikipedia).

We therefore obtain the complementary function

$$U_{\text{cf}} = C_1 J_0(s) + C_2 Y_0(s), \tag{3.12}$$

where C_1 and C_2 are constants of integration, and thus

$$\begin{aligned} U = U_{\text{cf}} + U_{\text{pi}} &= C_1 J_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}} r\right) + C_2 Y_0\left(\sqrt{\frac{-\mathrm{i}\omega}{\nu}} r\right) + \frac{G_0}{2\mathrm{i}\omega\rho} \\ &= C_1 J_0\left(\sqrt{-\mathrm{i}}\alpha\frac{r}{a}\right) + C_2 Y_0\left(\sqrt{-\mathrm{i}}\alpha\frac{r}{a}\right) + \frac{G_0}{2\mathrm{i}\omega\rho}, \end{aligned} \tag{3.13}$$

where the Womersley number α is a dimensionless parameter defined by $\alpha = a\sqrt{\frac{\omega}{\nu}}$.

Step 5 To find the constants C_1 and C_2 we apply the boundary conditions and regularity conditions at the origin $r = 0$:

- At $r = 0$: The function $Y_0(x)$ tends to infinity as $x \to 0$, so contributions from $Y_0((\sqrt{-\mathrm{i}\omega/\nu})r)$ are not allowed, meaning that C_2 must be set to zero.
- At $r = a$: The no-slip boundary condition is $u_z = 0$ at the wall for all times, which from Equation (3.7) means that U must also be zero at $r = a$. Substituting into Equation (3.13) gives

$$C_1 = -\frac{G_0}{2\mathrm{i}\omega\rho J_0(\sqrt{-\mathrm{i}}\alpha)}.$$

Hence

$$U = \frac{G_0}{2\mathrm{i}\omega\rho}\left(1 - \frac{J_0(\sqrt{-\mathrm{i}}\alpha r/a)}{J_0(\sqrt{-\mathrm{i}}\alpha)}\right), \tag{3.14}$$

and, substituting U into Equation (3.7), we get

$$u_z = -\frac{\mathrm{i}G_0}{2\omega\rho}\left(1 - \frac{J_0(\sqrt{-\mathrm{i}}\alpha r/a)}{J_0(\sqrt{-\mathrm{i}}\alpha)}\right)e^{\mathrm{i}\omega t}$$

$$= \frac{\mathrm{i}G_0}{2\omega\rho}\left[1 - \frac{J_0(\mathrm{i}^{3/2}\alpha\frac{r}{a})}{J_0(\mathrm{i}^{3/2}\alpha)}\right]e^{i\omega t} \qquad \text{with} \quad J_0(s) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!k!}\left(\frac{s}{2}\right)^{2k}, \tag{3.15}$$

3.2.3 Result and Extended Quantities

The Womersley solution Equation (3.15) is defined in the complex domain; but for simplicity, we only consider the real part to interpret its physical meaning.

1. The analytical solution u_z:

   $$u_z = \Re\left\{\frac{\mathrm{i}G_0}{2\omega\rho}\left[1 - \frac{J_0(\mathrm{i}^{3/2}\alpha\frac{r}{a})}{J_0(\mathrm{i}^{3/2}\alpha)}\right]e^{\mathrm{i}\omega t}\right\} \qquad \text{with} \quad J_0(s) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!k!}\left(\frac{s}{2}\right)^{2k},$$

   where J_0 is the Bessel function of the first-kind at 0^{th}-order.

2. The wall shear stress τ_{rz}:

   $$\tau_{rz} = \mu\frac{\partial u_z}{\partial r} = \mu\Re\left\{-\frac{a}{\mathrm{i}^{3/2}\alpha}\left(\frac{J_1(\mathrm{i}^{3/2}\alpha)}{J_0(\mathrm{i}^{3/2}\alpha)}\right)\frac{\partial p}{\partial z}\right\}, \quad \text{with} \quad J_1(s) = -\frac{\partial J_0(s)}{\partial s}.$$

3. The flow rate Q:

   $$Q(t) = \int_0^a 2\pi r u_z \mathrm{d}r = \Re\left\{-\frac{\pi a^4}{\mathrm{i}\mu\alpha^2}\left(1 - \frac{2J_1(i^{3/2}\alpha)}{\alpha\mathrm{i}^{3/2}J_0(\mathrm{i}^{3/2}\alpha)}\right)\frac{\partial p}{\partial z}\right\}.$$

The Womersley Number The Womersley number α is the ratio between unsteady inertia force and vthe iscous force.

- $\alpha \leq 1$: Quasi-steady, the velocity profile is scaled Poiseuille flow, mainly observed in the microvasculatures (*e.g.*, capillaries, venules);
- $\alpha > 1$: Oscillatory (*a.k.a.* plug flow), the velocity profile is balanced between viscous forces at the wall and inertial forces in the centre. Common in large arteries (*e.g.*, ascending aorta, carotid artery).

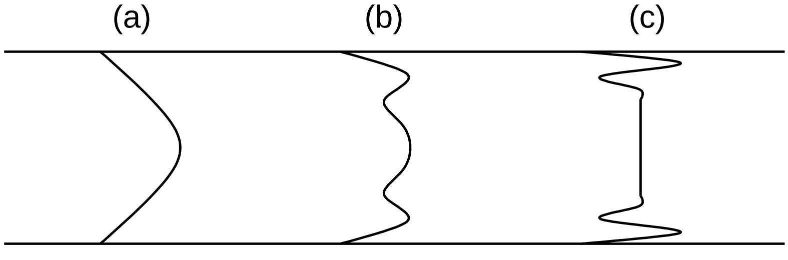


Figure 3.5: Snapshots of sketches of Womersley flow profiles for various Womersley numbers. (a) Low Womersley number (profile is Poiseuille flow multiplied by a time-dependent factor), (b) intermediate Womersley number, (c) high Womersley number (profile is flat across the interior with a boundary layer in which the flow oscillates rapidly).

Application of Womersley solution How do we use the Womersley solution to model a more complex flow? Since the Womersley flow assumes no convective acceleration, the solution is linear. Therefore, we can break down any arbitrary flow waveform into the orthogonal bases, *e.g.*, the Fourier terms, with each mode being an independent Womersley solution, and superimpose all components to reconstruct the Womersley flow (Figure 3.6).

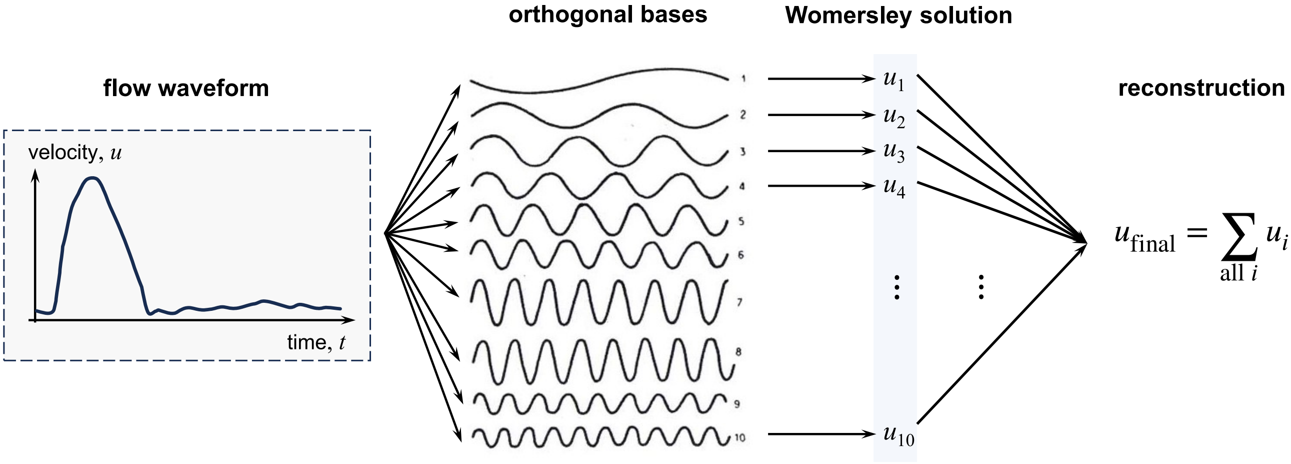


Figure 3.6: Schematic to decompose and reconstruct an arbitrary flow waveform using the Womersley solution.

Limitations of Womersley Solution

- No convective acceleration is considered – no tapering of the vessel, no vascular distension;
- The Womersley solution assumes flow in a straight pipe – effects such as bending are not considered;
- No entrance effects are considered, and flow is assumed to be fully developed;
- The Womersley solution assumes flow is laminar.

4 Turbulence and Energy Equations

4.1 Turbulence

The Reynolds number The Reynolds number[5], Re, is a dimensionless number that measures the ratio of the inertial force to the viscous force. For tube flow,

$$\mathrm{Re} = \frac{\rho U D}{\mu} = \frac{UD}{\nu} = \begin{cases} < 2000, & \text{laminar} \\ 2000 - 3000, & \text{transient} \\ > 3000, & \text{turbulent} \end{cases}, \tag{4.1}$$

where D is the pipe diameter, and U is the average velocity.

Comments

Although a high Reynolds number is often correlated with turbulence, the true cause is the dominance of inertial forces over viscous forces - high Re merely reflects (but does not by itself cause) that dominance.

Turbulence characteristics

- Random variation of fluid properties (*e.g.*, pressure and velocities) in time and space. Each property (*e.g.*, velocity, pressure, kinetic energy) has a specific continuous energy spectrum which drops to zero at high wave numbers (*e.g.*, Figure 4.9 for turbulent kinetic energy);
- Eddies or fluid structures spanning a wide range of length scales, which interact and transfer energy from large scales to progressively smaller ones; this process is known as the energy cascade (Figure 4.1);
- Self-sustaining motion – once triggered, turbulent flow can maintain itself by producing new eddies to replace those lost to viscous dissipation;
- Mixing – rapid convection of mass, momentum and energy, much stronger than laminar flows (Figure 4.2).

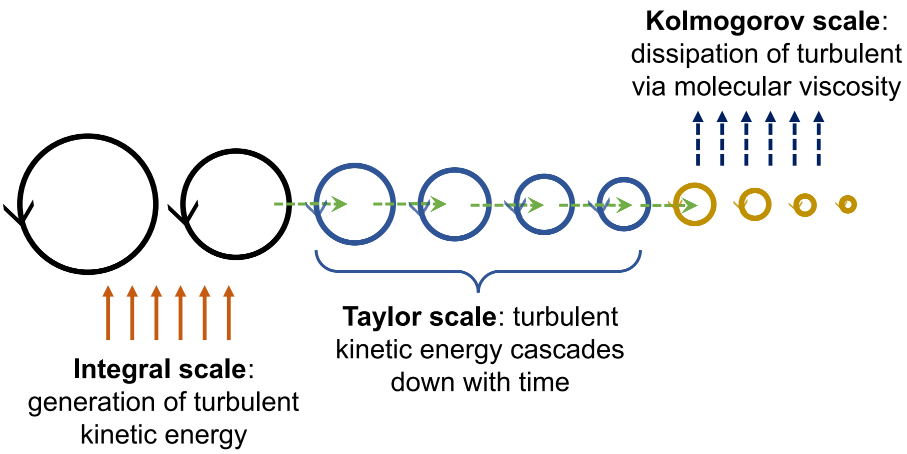


Figure 4.1: Cascade of turbulence kinetic energy. The turbulence kinetic energy is generated on the integral scale (large eddies) and dissipated on the Kolmogorov scale (small eddies).

[5] named after Osborne Reynolds (1842-1912).

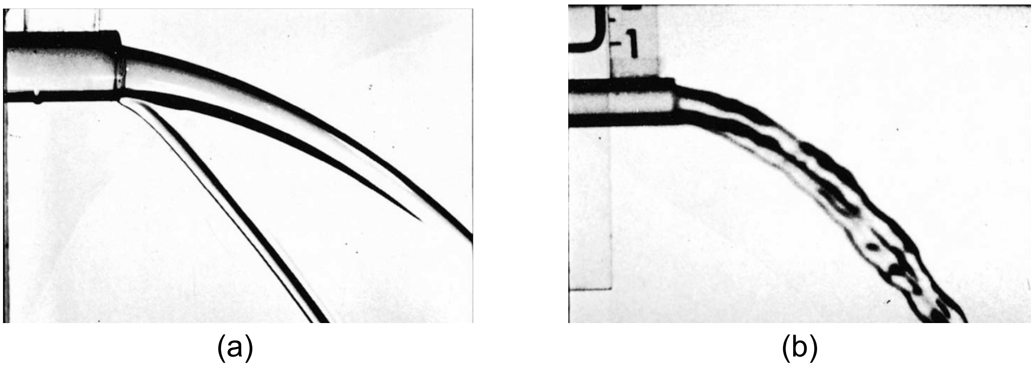


Figure 4.2: (a) High viscosity, low Re, the laminar flow behaves quasi-steady; (b) low-viscosity, high Re, turbulent flow. (National Committee for Fluid Mechanics Films)

Reynolds averaging and turbulent shear stress Turbulence cannot be deterministically predicted. It can only be *characterised* in a *statistical* manner by decomposing a certain flow quantity (*e.g.*, velocity) into the mean and standard deviation (fluctuation) components.

$$u(t) = \bar{u} + u'(t), \tag{4.2}$$

where

- Average velocity: $\bar{u} = \dfrac{1}{T}\displaystyle\int_0^T u \, \mathrm{d}t$,
- Velocity fluctuation: $u'(t) = u(t) - \bar{u}$.

Figure 4.3 demonstrates the velocity components from a 1-D turbulent velocity profile (u component only).

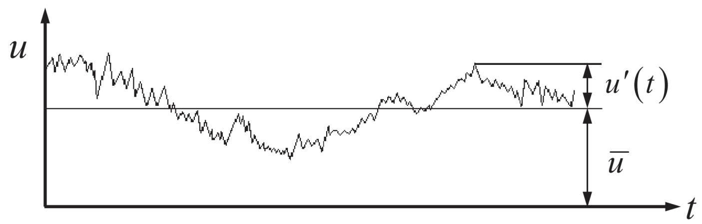


Figure 4.3: Turbulence velocity can be decomposed into the average component and the instantaneous velocity fluctuation.

With Equation 4.2, we can rewrite the x-momentum equation as (similar for y- and z-momentum equations)

$$\rho\frac{D\bar{u}}{Dt} = -\frac{\partial\bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial\bar{u}}{\partial x} - \rho\overline{u'u'}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z} - \rho\overline{u'w'}\right) + \rho f_x, \tag{4.3}$$

where the $\rho\overline{u_i'u_j'}$ terms are referred to as the Reynolds stresses or turbulent shear stresses (9 terms in total). The total shear stress is the sum of the laminar shear stress and turbulent shear stress:

$$\tau = \tau_{\mathrm{lam}} + \tau_{\mathrm{turb}} = \mu\frac{\partial\bar{u}_i}{\partial x_j} - \rho\overline{u_i'u_j'}. \tag{4.4}$$

Quantitative characteristics of turbulence

- <u>Turbulence intensity</u>, TI, measures how strong the velocity fluctuations are compared to the mean flow (Figure 4.4).

  For 1-D turbulent flow (u-component only):

$$\mathrm{TI} = \frac{\sqrt{\overline{u'^2}}}{\overline{u}}, \tag{4.5}$$

  where the numerator $\sqrt{\overline{u'^2}}$ is the root mean square (RMS) of velocity fluctuation.

  In the 3-D flow scenario (general form), TI can be expressed as

$$\mathrm{TI} = \frac{1}{\sqrt{\overline{u}_j\overline{u}_j}}\sqrt{\frac{1}{3}\overline{u'_i u'_i}} = \frac{1}{\sqrt{\overline{u}^2+\overline{v}^2+\overline{w}^2}}\sqrt{\frac{1}{3}(\overline{u'^2}+\overline{v'^2}+\overline{w'^2})}. \tag{4.6}$$

  Note that TI is dimensionless, and it is often reported as a percentage.

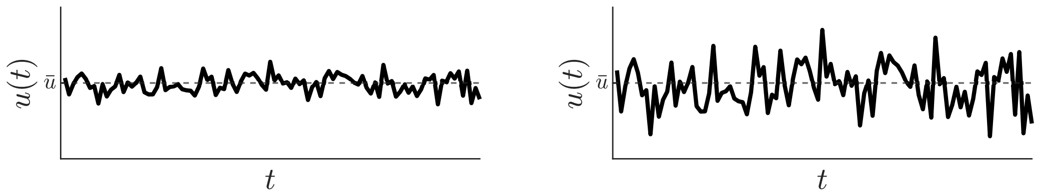


Figure 4.4: Two 1-D turbulent velocity profiles with identical mean velocity but different TI. Left: low TI; Right: high TI.

- <u>Turbulent kinetic energy</u>, k, quantifies how much kinetic energy is contained in the velocity fluctuations of a turbulent flow. Mathematically:

$$k = \frac{1}{2}\overline{u'_i u'_i} = \frac{1}{2}(\overline{u'^2}+\overline{v'^2}+\overline{w'^2}) \tag{4.7}$$

  Using Equation (4.6), TI can be written as

$$\mathrm{TI} = \frac{1}{\sqrt{\overline{u}_j\overline{u}_j}}\sqrt{\frac{2}{3}k}. \tag{4.8}$$

4.2 Turbulent Boundary Layer

What is Boundary Layer? The boundary layer (BL) is a thin layer of fluid in the vicinity of the wall in which the velocity rises from zero at the wall surface (no-slip) to the free-stream velocity (*i.e.*, U in Figure 4.5) away from the surface (along the y-direction in Figure 4.5). Outside the boundary layer, the mean flow velocity is U.

BLs can be laminar, transition, or turbulent. As depicted in Figure 4.5, the laminar BL is a smooth, thin layer; the turbulent BL contains swirls (eddies), and is generally thicker; while the transition boundary layer is in between.

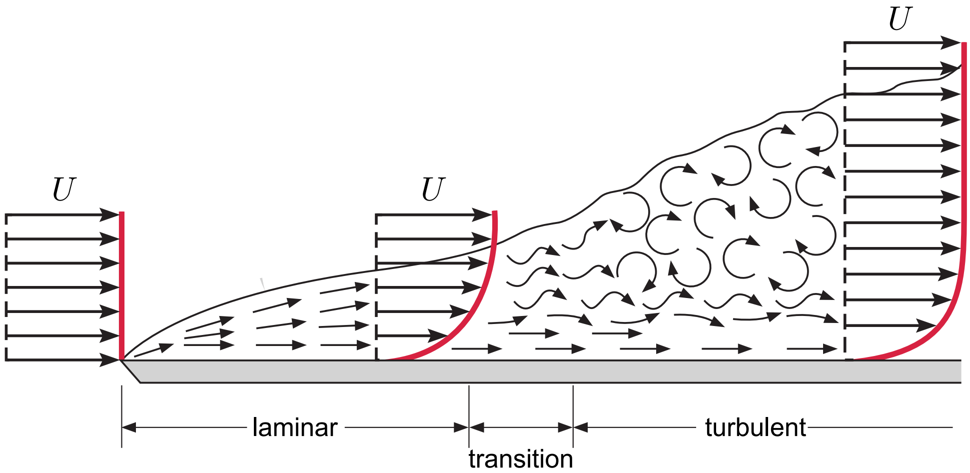


Figure 4.5: Velocity boundary layer development on a flat plate. (Incropera *et al.*)

Turbulent Boundary Layer As shown in Figure 4.6, the turbulent boundary layer can be further decomposed into 4 regions; from bottom to top, these are:

1. Viscous sublayer: the bottom layer closest to the wall, also known as the laminar layer, where the viscous effects dominate.
2. Buffer layer: on the top of the viscous sublayer, where the flow begins to feel the effect of turbulence, although laminar influence is still present.
3. Overlap layer: on the top of the buffer layer, also known as the inner layer, where it is gradually phasing out the near-wall region
4. Outer layer: the edge of the boundary layer, the freestream turbulence effects dominate.

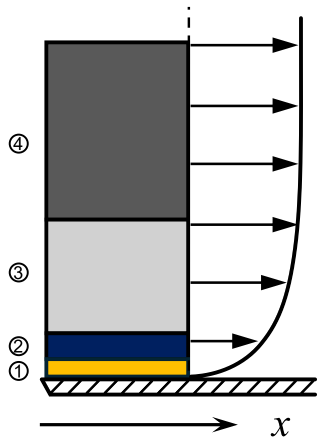


Figure 4.6: Four regions of the turbulent BL.

The viscous sublayer and the buffer layer are known as the near-wall region, which only comprises about 15% of the total turbulent boundary thickness.

The character of the flow within these regions can be very different, *e.g.*, the viscous effects are dominating in the viscous sublayer, but the opposite in the outer layer. How does the velocity profile vary within each region? The velocity-BL thickness relation is obtained by performing the dimensional analysis: define the dimensionless pairs $y^+ = y\frac{u^*}{\nu}$, $u^+ = \frac{\bar{u}}{u^*}$, where $u^* = \sqrt{\tau_w/\rho}$ is termed the friction velocity (*i.e.*, the velocity scale associated with the wall shear stress), ν is the kinematic viscosity, $\bar{u}$ is the mean turbulence velocity. The plot of u^+ versus y^+ is shown in Figure 4.7.

Layer	Range of y^+	u^+-y^+ Relation
viscous sublayer	$0 < y^+ < 5 \sim 8$	$u^+ \approx y^+$
buffer layer	$5 \sim 8 < y^+ < 30 \sim 70$	(blended)
overlap layer	$30 \sim 70 < y^+ < 10^4$	$u^+ = 1/\kappa \ln y^+ + B$
outer (wake) region	$y^+ > 10^4$	not strictly defined

(κ and B are both empirical constants.)

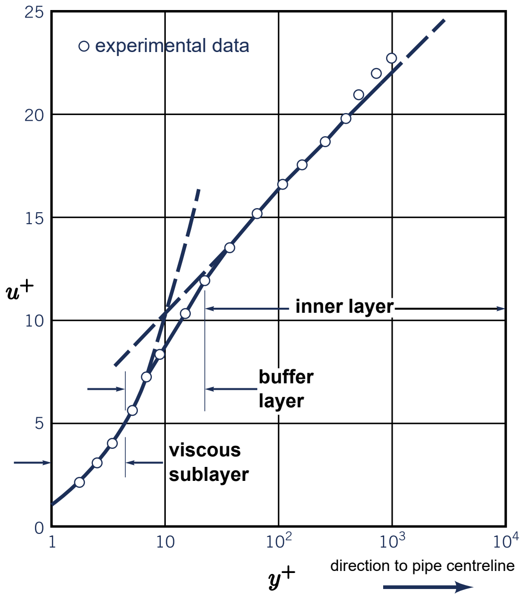


Figure 4.7: Typical structure of the turbulent velocity profile in a pipe. (Munson *et al.*)

For the outer region, one commonly adopted relation to describe the velocity profile is the power-law relation:

$$\frac{\bar{u}}{u_c} = \left(1 - \frac{r}{R}\right)^{1/n}, \tag{4.9}$$

where u_c is the centreline velocity; the value of n depends on Re, as the relation given out in Figure 4.8.

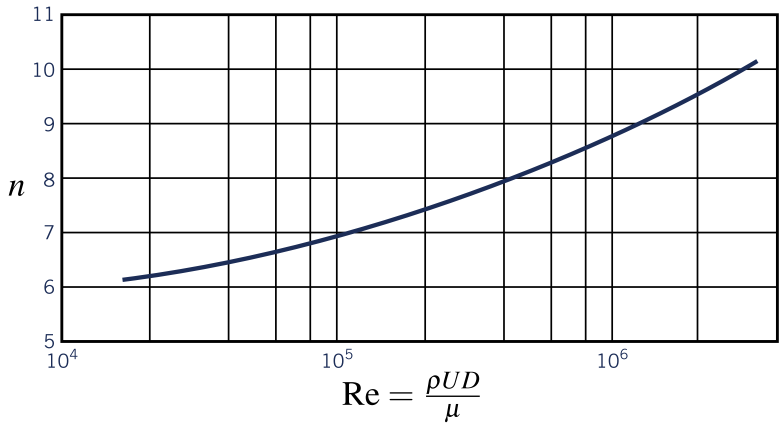


Figure 4.8: Exponent, n, for power-law velocity profiles. (Munson *et al.*)

4.3 Energy Cascade in Turbulent Flow

⚠ The mathematics in this section is optional and provided for further reading.

The progression, or breakdown of large eddies to small eddies, is referred to as the energy cascade. Let k $(= \frac{1}{2}\overline{u_i'u_i'})$ denotes the turbulence kinetic energy, ε denotes the dissipation rate of turbulence kinetic energy,

- **Integral scale**: the largest scale where the turbulence kinetic energy is generated. It can be related to the size of the system (*e.g.*, 10% ~ 20% of the pipe diameter). The integral length L and time τ_L scales are

$$L = \frac{k^{\frac{3}{2}}}{\varepsilon}, \qquad \tau_L = \frac{k}{\varepsilon}.$$

- **Kolmogorov scale**: the smallest scale which measures the size of the smallest eddies in the flow regime. This is where the turbulent kinetic energy is dissipated *via* the molecular viscosity. The Kolmogorov length η and time τ_η scales are

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}, \qquad \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}}.$$

- **Taylor scale**: the intermediate scale between the integral scale and Kolmogorov scale.

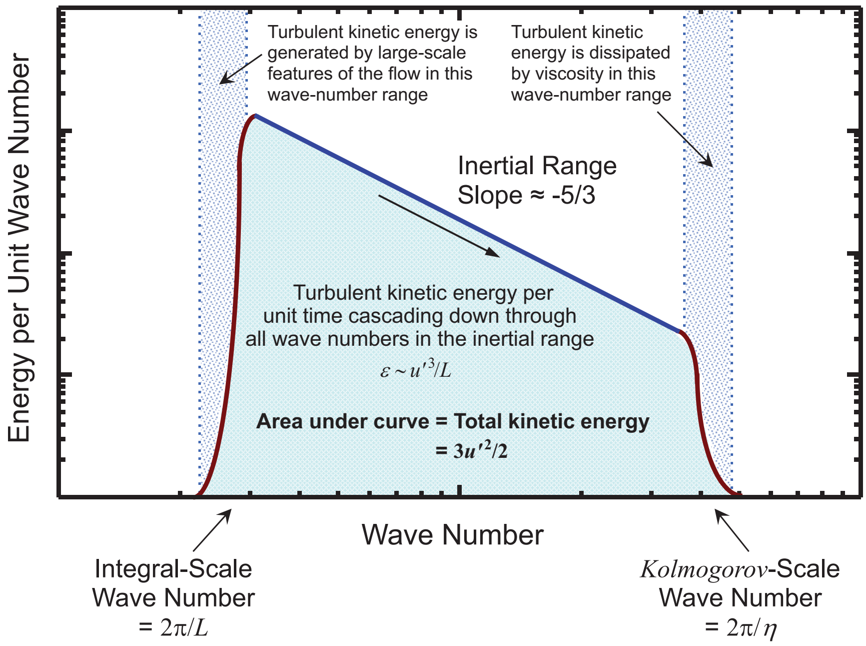


Figure 4.9: Turbulent kinetic energy cascade as a function of wave number (~ 1/size of eddies). (P. Aleiferis.)

Example

When milk is poured into coffee, the moving milk and the surrounding coffee collide and create an irregular motion that helps the two liquids mix quickly. Stirring with a spoon sets the whole cup of coffee into motion, forming a large swirl. This large motion then breaks into smaller swirls, which further spread the milk throughout the coffee, making the mixing fast and effective. This is a typical example of a turbulent energy cascade!

4.4 The Closure Problem

⚠ This topic is optional and provided for further reading.

One key problem that remains unanswered is that the velocity fluctuation $\overline{u_i'u_j'}$ that appears in the time-averaged form of the momentum equation is an unknown term. If we want to model or simulate turbulence computationally, these shear stress terms must be resolved. How do we model the turbulence then? We need an expression that bridges such unknown quantities to the known quantities, and this is referred to as the closure problem.

Joseph Valentin Boussinesq (1842-1929) proposed that, instead of finding the Reynolds stresses, one can alternatively find the turbulent viscosity. According to Boussinesq's theory, the Reynolds stress and the turbulent viscosity are linked through

$$\overline{u_i'u_j'} = \frac{2}{3}k\delta_{ij} - \nu_t \left[\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_j}\right],$$

where ν_t is the turbulent viscosity; note that ν_t is a property of the *turbulent flow*, not the fluid (whereas the kinematic viscosity ν is a property of fluid). This is known as the Boussinesq approximation.

The standard k-ϵ turbulence model (Launder *et. al.*, 1969), is a good example of turbulence modelling based on the Boussinesq approximation. In the solution procedure, the turbulent viscosity is calculated from the empirical relations. Subsequently, two additional transport equations are solved – one for turbulent kinetic energy k, one for turbulence kinetic energy dissipation rate ϵ, in addition to the continuity and momentum equations.

4.5 Bernoulli's Principle and Energy Equation

The Bernoulli's principle The Bernoulli's principle[6] states that, for an incompressible, inviscid fluid, if the flow is steady and laminar, the sum of pressure, kinetic and potential energy *per unit volume* is the same between two points lying on the same streamline:

$$p_1 + \frac{1}{2}\rho u_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho u_2^2 + \rho g h_2, \tag{4.10a}$$

Equation (4.10a) is known as the **pressure form** of the Bernoulli's principle (Figure 4.10).

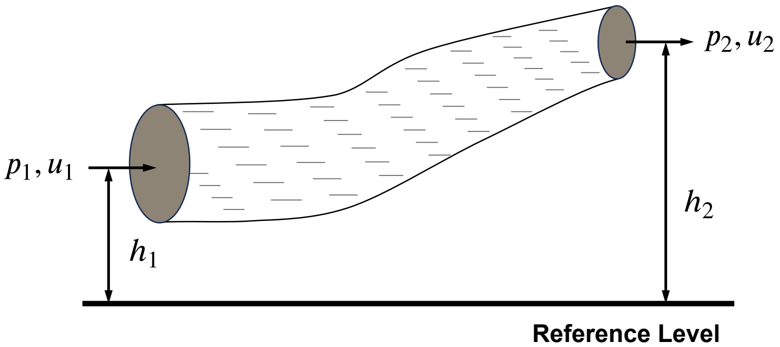


Figure 4.10: Schematic of steady incompressible flow through a conduit between two cross-sections, illustrating Bernoulli's principle.

Alternatively, dividing both sides of Equation (4.10a) by ρg yields the same principle in the **head form**:

$$\frac{p_1}{\rho g} + \frac{1}{2g}u_1^2 + h_1 = \frac{p_2}{\rho g} + \frac{1}{2g}u_2^2 + h_2, \tag{4.10b}$$

where $p/\rho g$ is known as the pressure head, $u^2/2g$ is the velocity head, and h is the potential (elevation) head.

Comments

The term "head" in the context above refers to the energy per unit weight of fluid. Historically, engineers measured energy as an equivalent column height of fluid – a 'head'.

The Bernoulli's principle can be interpreted as the perfect conservation of mechanical energy in *frictionless* flow.

Pipe flow energy equation Yet, in reality, there is no perfect conservation of the mechanical energy. Energy may dissipate by fluid friction with the rough walls, or due to a geometry change, such as expansion, contraction, or bending along a pipe. This necessitates the inclusion of an additional loss term to Equation (4.10), namely, the **energy equation in the head form**:

$$\frac{p_1}{\rho g} + \frac{1}{2g}u_1^2 + h_1 = \frac{p_2}{\rho g} + \frac{1}{2g}u_2^2 + h_2 + h_L, \tag{4.11a}$$

where h_L terms the total head loss, and can be further decomposed into the major head loss and minor head loss:

$$h_L = h_{L,\text{major}} + h_{L,\text{minor}}.$$

[6] named after Daniel Bernoulli (1700-1782).

However, the terms "major" and "minor" do not necessarily reflect the relative importance of each type of loss. The minor loss *can* be larger than the major loss.

Also note that one can simply convert Equation (4.11a) into its **pressure form**,

$$p_1 + \frac{1}{2}\rho u_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho u_2^2 + \rho g h_2 + \rho g h_L, \tag{4.11b}$$

where the term $\rho g h_L \equiv \Delta p_L$ is the pressure drop due to the head loss.

Major head loss The major head loss is the energy loss due to fluid friction, described by the **Darcy-Weisbach equation**,

$$h_{L,\text{major}} = f \frac{L}{D} \frac{U^2}{2g}, \tag{4.12}$$

where f is the (dimensionless) Darcy friction factor, L is the pipe length, D is the pipe diameter, U is the average velocity. For the fully developed, incompressible flow in a circular pipe, f is typically found as follows:

- If the flow is laminar, $f = 64/\mathrm{Re}$;
- If the flow is turbulent, f is obtained from the Moody diagram[7] (Figure 4.14), where the friction factor is related to the Reynolds number and the relative wall roughness of the pipe, $f\left(\mathrm{Re}, \frac{\varepsilon}{D}\right)$.

The major head loss leads to the pressure drop $\Delta p_{L,\mathrm{major}} = \rho g h_{L,\text{major}}$.

Comments

- For laminar flow, the pressure drop calculated with the friction factor $f = 64/\mathrm{Re}$ coincides with the Hagen-Poiseuille equation, where $\Delta p = \dfrac{8\mu L Q}{4\pi R^4}$.
- For turbulent flow, even for smooth pipes (*do not neglect this trace in Figure 4.14!*) the friction factor is not zero. This is a result of the non-slip boundary condition, which requires fluid to stick to the solid surface it flows over.
- For fully turbulent flow, *i.e.*, Re is sufficiently high, f is nearly independent to Re, but only depends on the surface roughness. This is because the viscous sublayer gets thinner as Re increases, hence, ε dominates any near-wall flow character.

Minor head loss Energy losses can also be associated with the geometrical features of a pipe. Examples are the bends and valves in a pipe system or changes in diameter (expansion or contraction) along the channel, which consequently alter the flow pattern and lead to the minor head loss. Such losses are commonly described using an empirically derived loss coefficient, K_L,

$$h_{L,\text{minor}} = K_L \frac{U^2}{2g}, \tag{4.13}$$

and by $\Delta p = \rho g h_{L,\text{minor}}$,

$$\Delta p = \frac{1}{2}\rho U^2 K_L. \tag{4.14}$$

We shall now explore a few types of minor losses:

1. **Pipes with a sudden, sharp-edged contraction or expansion**: K_L is related to the ratio of the cross-sectional areas of two sections (Figure 4.11).

[7] named after Lewis F. Moody (1880-1953).

Comments

In fact, by simple conservation principles in mass, momentum, and energy, the loss coefficient for the sudden expansion (SE) can be derived analytically:

$$K_{L,SE} = \left(1 - \frac{A_1}{A_2}\right)^2,$$

where A_1 denotes the cross-sectional area upstream of the area change, and A_2 denotes the cross-sectional area downstream of the area change. However, the loss coefficient for the sudden contraction (SC) could not be derived analytically, but fit experimentally, one possible option is

$$K_{L,SC} \approx 0.42\left(1 - \frac{A_2}{A_1}\right).$$

This prediction is valid up to the value $A_2/A_1 = 0.76$.

2. **Diffusers** (a pipe device with a gradual expansion in diameter, used to decelerate the fluid flow): K_L is related to the ratio of the cross-sectional areas of two sections, as well as the cone angle of expansion (Figure 4.12).

Comments

The head loss in a diffuser arises from wall shear stress and from incomplete pressure recovery (how efficiently the kinetic energy is converted to the static pressure) caused by flow separation.

By Figure 4.12, K_L drops due to reduced friction when $0° < \theta < 15°$ – where a diffuser is considered efficient; K_L then increases sharply due to the flow separation, when the adverse pressure gradients intensify; in this regime, the loss coefficient becomes comparable to that of a sudden expansion.

3. **Pipes with a bend**: K_L is linked to the ratio between the radius of pipe curvature and pipe diameter, as well as the relative roughness values (Figure 4.13).

Comments

Note that Figure 4.11, Figure 4.12, and Figure 4.13 only apply to turbulent flow – You need to calculate Re before reading values from the chart!

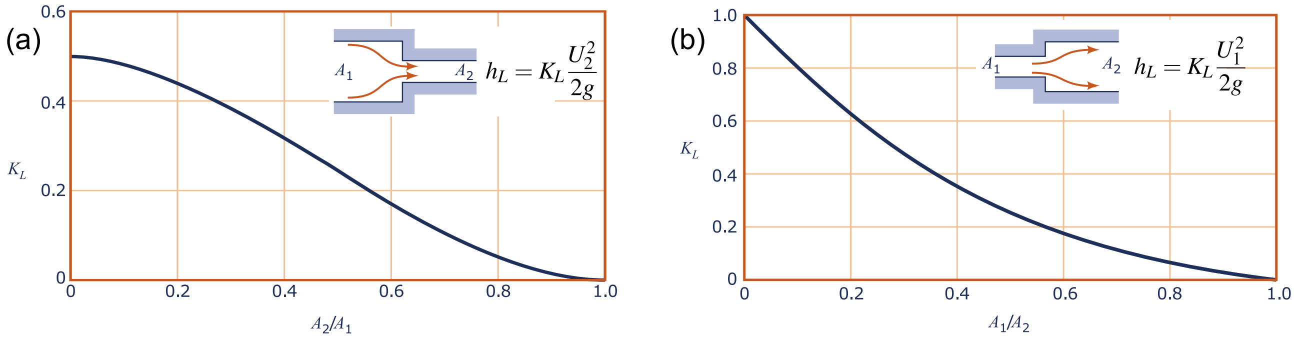


Figure 4.11: Loss coefficient for the sharp-edged (a) contraction, (b) expansion. (Munson *et al.*)

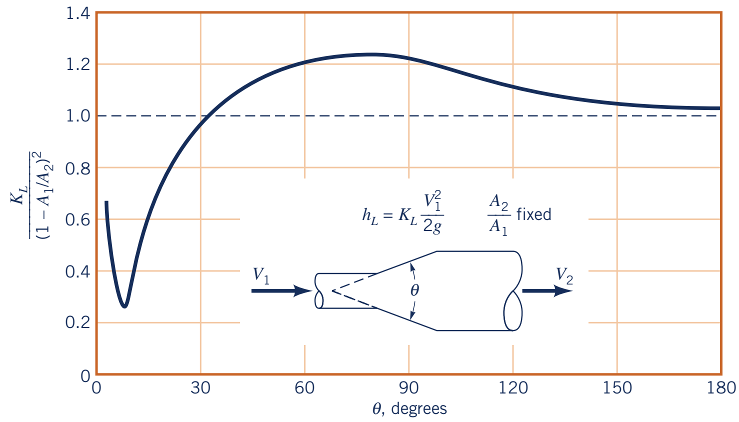


Figure 4.12: Loss coefficient for a typical diffuser. (Munson *et al.*)

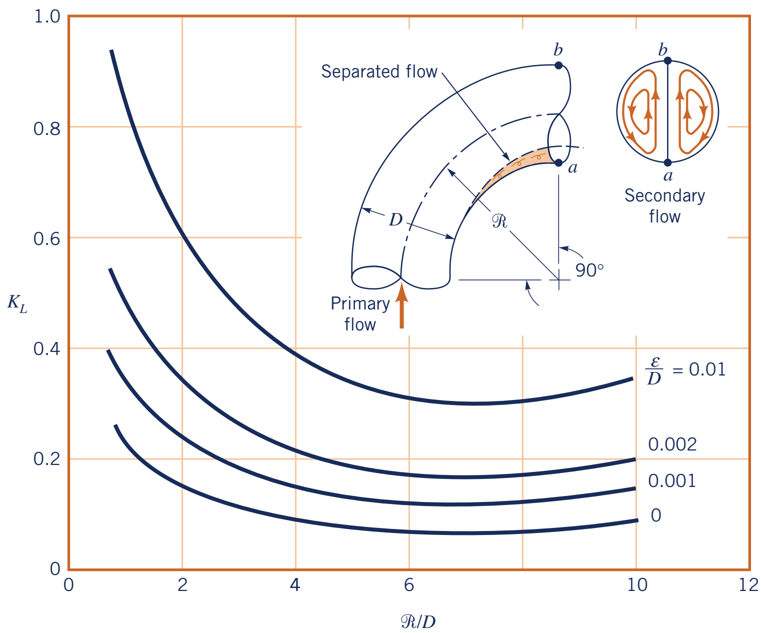


Figure 4.13: Loss coefficient for a pipe with a 90° bend. (Munson *et al.*)

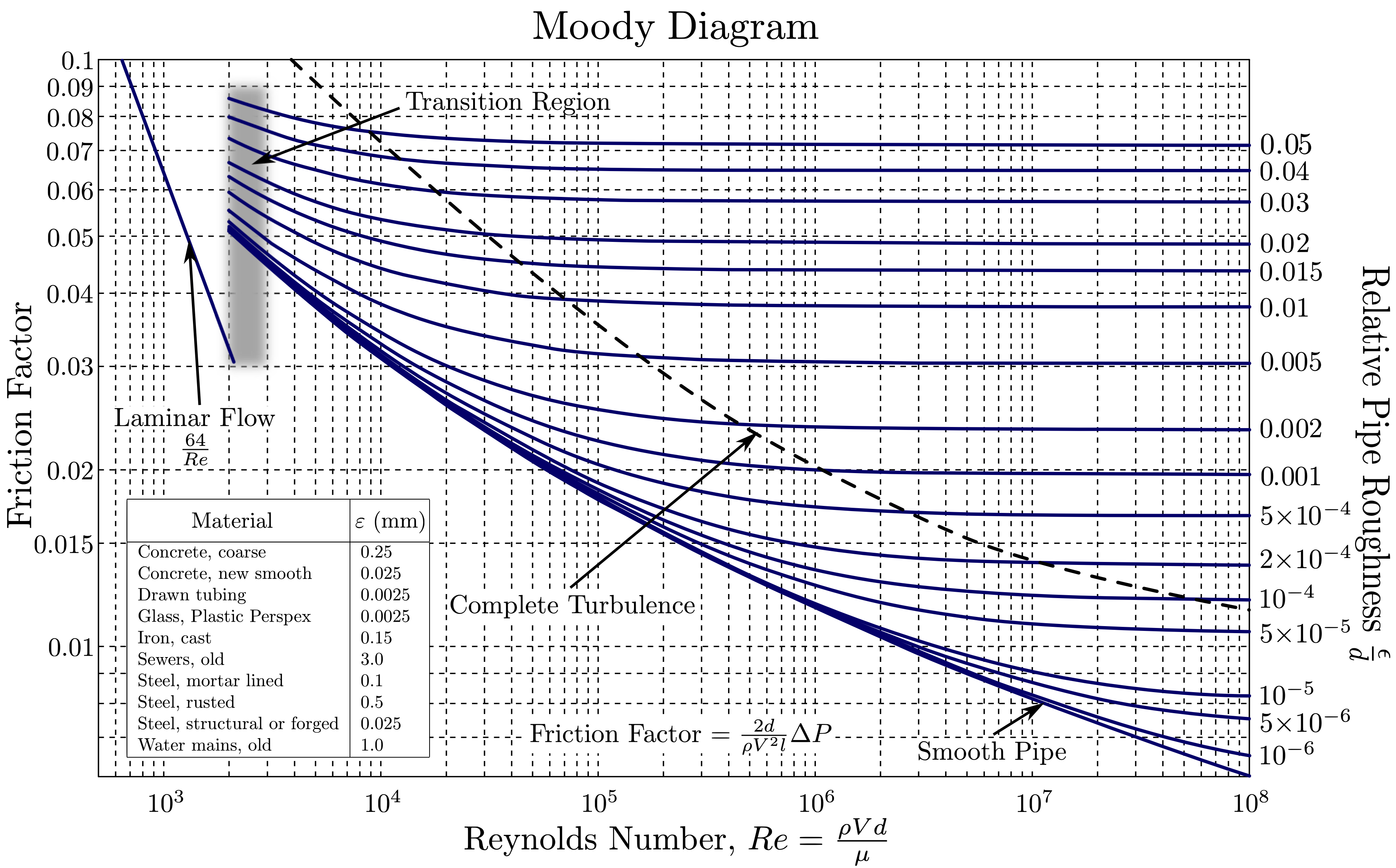


Figure 4.14: Moody chart. (Wikipedia)

5 Problems Involving Scaling

5.1 Nondimensionalisation and Buckingham-Π Theorem

In fluid problems, the aim is to express a physical quantity in terms of the relevant input variables and parameters in the problem (*e.g.* lengths and geometry, material properties of the fluid, velocities, ...). That is, we want to find a quantity Y in terms of k input variables $\{y_1, y_2, y_3, \dots, y_k\}$.

$$Y = f(y_1, y_2, \dots, y_k). \tag{5.1}$$

The Buckingham-Π theorem states that, Equation (5.1) can be nondimensionalised into the form

$$Y^* = f^*(y_1^*, y_2^*, \dots, y_r^*), \tag{5.2}$$

where Y^* denotes the nondimensionalised form of Y; y_1^*, y_2^*, ..., y_r^* are the essential (minimum) reference dimensions required to describe y_1, y_2, ..., y_k. Note that $r \le k$, consequently, there will be $(k - r)$ *independent* dimensionless products (Π groups).

$$\Pi_1 = \phi(\Pi_2, \Pi_3, ...\Pi_{k-r}). \tag{5.3}$$

Comments

Nondimensionalising is a powerful tool in fluid mechanics for two main reasons:

- **Efficiency:** The number of parameters in the problem decreases. Thus each set of parameters in the nondimensionalised system corresponds to a whole family of parameter values in the dimensional system. Thus every experimental result or numerical simulation enables us to understand the system for a whole range of sets of parameter values (rather than just one set of parameter values). In turn this either decreases the number of experiments or simulations that need to be performed and/or increases our level of understanding about the problem.

- **Simplification:** If the appropriate scalings are adopted, it often happens that one of the nondimensional parameters is particularly large or particularly small. This usually means that certain terms in the equations are dominant and others may be neglected (or at least assumed to be small). Neglecting unimportant terms can mean we can make significantly more progress in our analysis of the problem than would otherwise be possible. We will see some examples of this in the rest of this section.

Reference Dimensions The dimensions of all the quantities y_1, ..., y_k are written as combinations of reference dimensions, y_1^*, ..., y_r^*. The most common reference dimensions are $[M]$ for mass, $[L]$ for length, and $[T]$ for time.

For example, the dimensions for the velocity can be written as $[LT^{-1}]$ (*cf.* length/time). Alternatively, in some cases (e.g., In subsection 5.2), the dimensions for the velocity is written as $[U]$ which denotes the velocity scale.

The dimensions of the common quantities are summarised in the following table.

Quantity	Symbol	Dimensions	Quantity	Symbol	Dimensions
Acceleration	a	$[L^1T^{-2}]$	Surface tension	σ_s	$[M^1T^{-2}]$
Angle	θ, ϕ, etc.	1 (none)	Velocity	U	$[L^1T^{-1}]$
Density	ρ	$[M^1L^{-3}]$	Viscosity	μ	$[M^1L^{-1}T^{-1}]$
Force	F	$[M^1L^1T^{-2}]$	Volume flow rate	Q	$[L^3T^{-1}]$
Frequency	f	$[T^{-1}]$	Pressure	p	$[M^1L^{-1}T^{-2}]$

Example: Applying The Buckingham-Π theorem

Objective Perform the dimensional analysis of the scenario where the pressure drops per unit length along a smooth pipe.

Step 1 List all relevant variables in the objective equation to be nondimensionalised. Here,

$$\Delta p_l = f(D, \rho, \mu, U),$$

where the pressure drop Δp_l is a function of the pipe diameter D, the density ρ, the (dynamic) viscosity μ, and velocity U.

Step 2 List the dimensions of the variables. Let $[M]$ denotes the dimension of mass, $[L]$ denotes the dimension of length, $[T]$ denotes the dimension of time,

$$\begin{aligned} \Delta p_l &\doteq [ML^{-1}T^{-2}], & \mu &\doteq [ML^{-1}T^{-1}] \\ D &\doteq [L], & U &\doteq [LT^{-1}] \\ \rho &\doteq [ML^{-3}] \end{aligned}$$

There are $k = 5$ variables and $r = 3$ reference dimensions, we conclude there will be $k - r = 2$ dimensionless groups.

Step 3 Suppose the first group involves Δp_l, ρ, U and D. Let a, b, c, d denote 4 constants to be determined,

$$D^a \rho^b V^c \Delta p_l^d \quad \Longrightarrow \quad \boxed{[L]^a [ML^{-3}]^b [LT^{-1}]^c [ML^{-1}T^{-2}]^d \doteq [L]^0 [F]^0 [T]^0}.$$

Balance of $[M]$, $[L]$, $[T]$ would give the simultaneous equations

$$\begin{aligned} \text{(mass)} \quad & b + d = 0, \\ \text{(length)} \quad & a - 3b + c - d = 0, \\ \text{(time)} \quad & -c - 2d = 0. \end{aligned}$$

(3 equations with 4 unknowns $\Rightarrow$ the equation system is underdetermined, we will not be able to explicitly solve the numerical values of 4 parameters, but at least we will know the relations between a, b, c, d.)

resulting in the following relations: $a = 0$, $b = -d$, $c = -2d$. Hence, with $d = -1$, $\longrightarrow$ $a = 0$, $b = 1$, $c = 2$,

$$D^0 \rho^1 U^2 \Delta p_l^{-1} \equiv \left(\frac{\rho U^2}{\Delta p_l}\right) \quad \text{is dimensionless,} \quad \Longrightarrow \quad \boxed{\Pi_1 = \left(\frac{\rho U^2}{\Delta p_l}\right)}.$$

(Although we *supposed* that D might get involved in the first Π group, but by $a = 0$, Π_1 is invariant of D.)

Step 4 Similarly, the second term involves μ, follow the same rule, this yields $\boxed{\Pi_2 = \dfrac{\mu}{\rho D U}}$, which is $1/\mathrm{Re}$.

Step 5 Hence, we can express the result of the dimensional analysis as

$$\frac{\rho U^2}{\Delta p_l} = \phi\left(\frac{\mu}{\rho D U}\right).$$

It expresses the idea that the dimensionless pressure drop solely depends on the ratio of viscous force to the inertial force. In other words, the flow-induced pressure loss scales with Re.

Comments

- By $\Delta p_l = f(D, \rho, \mu, U)$, the conventional way to investigate Δp_l with each parameter requires holding the others constant; *i.e.*, when studying the Δp_l-D relation, ρ, μ, U must be kept constant. If we make one plot for each parameter, there will be 4 plots in total.
- By Buckingham-Π theorem, $\dfrac{\rho U^2}{\Delta p_l} = \phi\left(\dfrac{\mu}{\rho D U}\right)$, we know the flow-induced pressure loss *solely* scales with Re, and all above information can be encapsulated into 1 plot.
- This example shows how the dimensional analysis reduces the complexity, cost, and time required to determine the relationship between a physical quantity and the other variables.
- However, we should note that each dimensionless group must hold a meaningful physical interpretation.

Summary of common variables and dimensionless groups in fluid mechanics:

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, L; Density, ρ; Frequency of oscillating flow, ω; Pressure, p; Speed of sound, c; Surface tension, σ_s ; Velocity, U.

Dimensionless groups	Name	Interpretation	Types of Applications
$\rho U L/\mu$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$U/\sqrt{gL}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$p/\rho U^2$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
U/c	Mach number, Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\omega L/U$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\rho U^2 L/\sigma_s$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

5.2 The Dimensionless Navier-Stokes Equations

In this section, we investigate common ways of nondimensionalising problems in biological fluid mechanics. Assuming the body forces are insignificant, the Navier-Stokes and continuity equations (2.7) and (2.3) are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{u}, \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{5.4}$$

It is often the case that a problem in fluid mechanics has a typical (characteristic) length scale, L, and a typical (characteristic) fluid speed, U. We naturally obtain the following dimensionless variables:

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad t^* = \frac{t}{L/U}, \quad p^* = \frac{p}{P_0}, \tag{5.5}$$

where P_0 is the characteristic pressure scale to be determined later. Note that we use superscript * to indicate corresponding dimensionless quantities. On substituting these into equations 5.4 we obtain

$$\frac{U^2}{L}\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \boldsymbol{\nabla}^*)\mathbf{u}^*\right) = -\frac{P_0}{\rho L}\boldsymbol{\nabla}^* p^* + \frac{\nu U}{L^2}\nabla^{*2}\mathbf{u}^*, \qquad \frac{U}{L}\boldsymbol{\nabla}^* \cdot \mathbf{u}^* = 0. \tag{5.6}$$

Note that the differential operator $\boldsymbol{\nabla}$ must be nondimensionalised and also the derivative with respect to time. Dividing equations (5.6) respectively by the factor in front of the viscous term and by U/L, they become

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \boldsymbol{\nabla}^*)\mathbf{u}^*\right) = -\frac{P_0}{(\mu U/L)}\boldsymbol{\nabla}^* p^* + \nabla^{*2}\mathbf{u}^*, \qquad \boldsymbol{\nabla}^* \cdot \mathbf{u}^* = 0. \tag{5.7}$$

where $\mathrm{Re} = UL/\nu$ is the Reynolds number.

We still need to choose the pressure scale P_0, as there is no natural scaling for pressure. Usually, we assume that the pressure gradient plays an important role in the problem, meaning that it is of the same order of magnitude as the largest term in the equation; *i.e.*, it depends on whether viscous effects or inertial effects are more dominant.

$$P_0 = \frac{\mu U}{L}\max\left(1, \mathrm{Re}\right). \tag{5.8}$$

In the case that the Reynolds number is very large or very small, this leads to considerable simplification of the equations, which will be discussed in the next sections.

Comments

Since all the starred variables have an order of magnitude 1, $O(1)$, we can see from (5.7) that a physical interpretation of the Reynolds number equals the ratio of the typical acceleration of fluid particles to the typical viscous force per unit mass.

Scaling for low-Reynolds-number flows ($\mathrm{Re} \ll 1$) In this case (5.8) gives $P_0 = \mu U/L$, and Equation (5.7) becomes

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \boldsymbol{\nabla}^*)\mathbf{u}^*\right) = -\boldsymbol{\nabla}^* p^* + \nabla^{*2}\mathbf{u}^*, \qquad \boldsymbol{\nabla}^* \cdot \mathbf{u}^* = 0. \tag{5.9}$$

Since the Reynolds number is very small, to leading order we may neglect the L.H.S. of the equation, and the Navier-Stokes equation reduces to the Stokes equation:

$$\nabla^{*2}\mathbf{u}^* = \boldsymbol{\nabla}^* p^*, \qquad \boldsymbol{\nabla}^* \cdot \mathbf{u}^* = 0. \tag{5.10}$$

Comments

- The Stokes equation is much simpler to solve than the Navier-Stokes equation, primarily because it is linear.
- The Stokes equation can be rewritten using in terms of the vorticity: $\nabla^2\boldsymbol{\omega} = 0$ - This is often simpler to find the general solution! The derivation follows:

  1. Redimensionalise the Stokes equation, yielding

  $$\mu\nabla^2\mathbf{u} = \boldsymbol{\nabla}p, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

  2. Define the vorticity as $\boldsymbol{\omega} = \nabla\times\mathbf{u}$

  $$\mu\nabla^2\mathbf{u} = -\mu\nabla\times\boldsymbol{\omega} \qquad \text{due to} \qquad \nabla\times\boldsymbol{\omega} = \nabla\times(\nabla\times\mathbf{u}) = \cancel{\nabla\cdot\mathbf{u}} - \nabla^2\mathbf{u}.$$

  3. Further, take the curl of $\mu\nabla^2\mathbf{u} = \nabla p$:

  $$\underbrace{\nabla\times\nabla p}_{\text{"curl of grad is zero"}} = \nabla\times(\mu\nabla^2\mathbf{u}) \implies \begin{aligned} 0 &= -\mu\nabla\times(\nabla\times\boldsymbol{\omega}) \\ 0 &= -\mu[\underbrace{\nabla(\nabla\cdot\boldsymbol{\omega}) - \nabla^2\boldsymbol{\omega}}_{\text{by: } \nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}}] \\ 0 &= -\mu[\underbrace{\nabla(\nabla\cdot\nabla\times\mathbf{u})}_{\text{"div of curl is zero"}} - \nabla^2\boldsymbol{\omega}] \\ 0 &= \nabla^2\boldsymbol{\omega}. \end{aligned}$$

which can give a method to solve the problem (as discussed in Section 5.5).

Scaling for high-Reynolds-number flows ($\mathrm{Re} \gg 1$) In this case, (5.8) gives $P_0 = \mathrm{Re}\cdot\mu U/L = \rho U^2$, and Equation (5.7) becomes

$$\frac{\partial\mathbf{u}^*}{\partial t^*} + (\mathbf{u}^*\cdot\boldsymbol{\nabla}^*)\mathbf{u}^* = -\boldsymbol{\nabla}^*p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^*, \qquad \boldsymbol{\nabla}^*\cdot\mathbf{u}^* = 0. \tag{5.11}$$

Since the Reynolds number is very large, to leading order we may neglect the viscous term, and the Navier-Stokes equation reduces to

$$\frac{\partial\mathbf{u}^*}{\partial t^*} + (\mathbf{u}^*\cdot\boldsymbol{\nabla}^*)\mathbf{u}^* = -\boldsymbol{\nabla}^*p^*, \qquad \boldsymbol{\nabla}^*\cdot\mathbf{u}^* = 0. \tag{5.12}$$

Thus, to leading order, the fluid behaves like an inviscid fluid.

Comments

- Equations (5.12) represents a different type of differential equation from the scaled Navier–Stokes equations (5.11), since the viscous term in (5.11) is the term containing the highest order derivatives. In the absence of the viscous term, it is not possible to impose the usual number of boundary conditions. It is usual merely to impose no-penetration (instead of full no-slip) boundary conditions (which is the same as what we would do with an inviscid fluid). Thus we require that no fluid flows through an impermeable wall, that is $\mathbf{u}_r^*\cdot\mathbf{n} = 0$, where $\mathbf{u}_r$ is the fluid velocity minus the wall velocity and $\mathbf{n}$ is the normal vector to the wall.

- In practice, a thin boundary layer develops near the wall:
  - At the wall, the fluid velocity equals the velocity of the wall (which is zero in the case of a fixed wall), due to the no-slip boundary conditions.
  - At the edge of the boundary layer, the fluid velocity equals the bulk velocity (that is the velocity we calculated using the inviscid approximation with the no-penetration boundary conditions).
  - Within the boundary layer there are typically large gradients in the fluid velocity as it changes between these values over a short distance.

  If we require the details of the flow within the boundary layer (for example if we need to calculate the shear stress that the fluid flow induces on the wall), we can use the following method:

  - Away from the walls, we solve the simplified Equation (5.12).
  - Within the boundary layers, we cannot neglect the viscous terms, but we can make scalings that simplify the equations considerably. This is because the boundary layer is very thin, so we assume that the coordinate variable perpendicular to the wall is much smaller than the coordinate parallel to the wall. The equations to be solved within the boundary layer will be derived in Section 5.4. We then apply no-slip boundary conditions at the wall, and matching conditions at the edge of the boundary layer, which tells us both the width of the boundary layer and the flow profile within it.

In the rest of this section, we consider a few particular problems in fluid mechanics in which using dimensional analysis enables us to either find a solution or to make significant simplifications of the governing equations. In particular:

- **Lubrication flows** (Section 5.3): occurs when the domain of the fluid is long and thin, such that velocity gradients normal to the boundaries dominate those in the streamwise direction, and the flow is governed primarily by a balance between viscous stresses and pressure gradients.
- **Boundary layer flows** (Section 5.4): occur when $\mathrm{Re} \gg 1$, where viscous effects are confined to thin regions in the vicinity of the boundary. Outside this viscous dominant region, the flow is governed by the inertial effects.
- **Stokes flows** (Section 5.5): occur when $\mathrm{Re} \ll 1$, where the inertial effects may be neglected relative to viscous forces, so that the flow is governed by a balance between viscous stresses and pressure gradients.

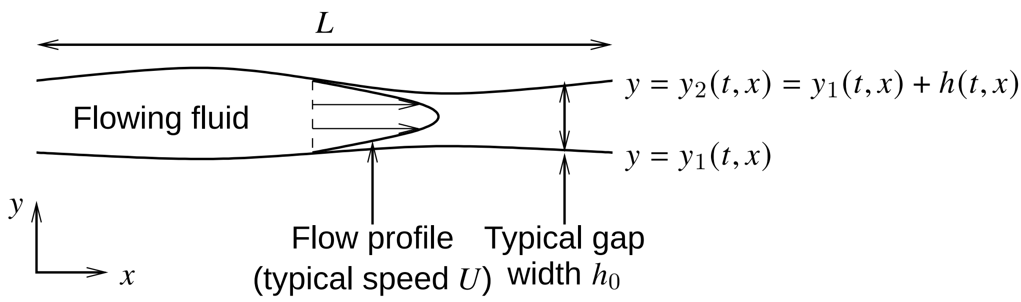
5.3 Lubrication Theory

⚠ This topic is optional and provided for further reading.

Motivation The lubrication theory is a technique used to find an approximate solution when the domain of the fluid is long and thin. We use it because it results in a considerable simplification of the Navier-Stokes equations. The basic assumption is that fluid flow properties vary much more quickly across the layer than along the layer.

One motivation for this type of analysis is that it can be very difficult and costly to simulate long and thin regions numerically because flow properties change so rapidly across the layer. However, the thinner the region becomes, the more accurate the approximations presented in this section become, and thus, it becomes a method of choice in extreme cases.

Derivation



Step 1: Choosing the characteristic (scaled) variables For simplicity, we work in two dimensions (x, y), with the following assumptions and constraints:

- The fluid flows in a channel whose typical width h_0 in the y-direction is much smaller than the length L in the x-direction; their ratio $\varepsilon = h_0/L \ll 1$;
- The side walls are at $y = y_1(t, x)$ and $y = y_2(t, x)$, with the channel height $h = y_2 - y_1$ and h_0 being a typical value of the function h;
- U is the characteristic velocity along the channel in the x-direction.

Therefore, the scaled variables are chosen as

$$x = Lx^*, \quad y = h_0 y^*, \quad t = \frac{L}{U}t^*, \quad p = p_0 p^*.$$

The only exception we will further discuss below is the pressure scale p_0, as there is no natural scaling to the pressure term.

Since U is the characteristic velocity along the x-direction, meaning that typical changes in the x-component of velocity are of order U and hence a typical value of $\partial u/\partial x$ is of order U/L. By the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

the order of magnitude of $\dfrac{\partial u}{\partial x}$ must be balanced by the order of magnitude of $\dfrac{\partial v}{\partial y}$ - hence, a typical value of v is of order $h_0 U/L$. Thus we set

$$u = Uu^*, \quad v = \frac{h_0 U}{L} v^*.$$

Step 2: nondimensionalise the continuity equation We substitute the scaled variables into the continuity and Navier-Stokes equations to get the system in terms of the nondimensional variables. The continuity equation gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{h_0 U}{h_0 L}\frac{\partial v^*}{\partial y^*} = 0 \quad \Rightarrow \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \tag{5.13}$$

Step 3: nondimensionalise the x-component of the N-S equation The x-component of the N-S equation becomes

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$\Rightarrow \quad \rho\left(\frac{U^2}{L}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{h_0 U^2}{h_0 L}v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{p_0}{L}\frac{\partial p^*}{\partial x^*} + \mu\left(\frac{U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{U}{h_0^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right).$$

Dividing by the coefficient of the viscous term $\partial^2 u^*/\partial y^{*2}$, that is, dividing by $\mu U/h_0^2$, this becomes

$$\varepsilon^2 \mathrm{Re}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{h_0^2 p_0}{\mu U L}\frac{\partial p^*}{\partial x^*} + \varepsilon^2\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}},$$

where $\mathrm{Re} = \rho U L/\mu$ is the Reynolds number of the flow. We call the parameter $\varepsilon^2\mathrm{Re}$ the reduced Reynolds number associated with the problem, and typically in lubrication theory we assume it is small $\varepsilon^2\mathrm{Re} \ll 1$. If we also neglect the term whose coefficient is ε^2, which is expected to be small, we obtain

$$0 = -\frac{h_0^2 p_0}{\mu U L}\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{5.14}$$

Step 4: nondimensionalise the y-component of the N-S equation Similarly, we nondimensionalise the y-component of the N-S equations as

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$
$$\Rightarrow \quad \rho\left(\frac{h_0 U^2}{L^2}\frac{\partial u^*}{\partial t^*} + \frac{h_0 U^2}{L^2}u^*\frac{\partial u^*}{\partial x^*} + \frac{h_0^2 U^2}{h_0 L^2}v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{p_0}{h_0}\frac{\partial p^*}{\partial x^*} + \mu\left(\frac{h_0 U}{L^3}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{h_0 U}{h_0^2 L}\frac{\partial^2 u^*}{\partial y^{*2}}\right).$$

If we divide by the same coefficient (for comparison with the x-component), $\mu U/h_0^2$, this becomes

$$\varepsilon^3 \mathrm{Re}\left(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{h_0 p_0}{\mu U}\frac{\partial p^*}{\partial y^*} + \varepsilon^3\frac{\partial^2 u^*}{\partial x^{*2}} + \varepsilon\frac{\partial^2 u^*}{\partial y^{*2}}.$$

We can neglect both the inertial terms, whose coefficient is $\varepsilon^3\mathrm{Re}$, and the term with coefficient ε^3, as these are both small compared to the term whose coefficient is ε. This gives

$$0 = -\frac{h_0 p_0}{\mu U}\frac{\partial p^*}{\partial y^*} + \varepsilon\frac{\partial^2 v^*}{\partial y^{*2}}. \tag{5.15}$$

Step 5: Choosing the the pressure scale p_0 We have reduced the problem to the three nondimensional equations (5.13), (5.14) and (5.15), and still need to choose the pressure scale p_0. Inspecting Equation (5.14) and (5.15), the scales that give balances in these equations are when p_0 is of order $\mu U L/h_0^2$ (balance in (5.14)) or order $\mu U/h_0$ (balance in (5.15)). Thus there are five categories for the choice of the scale for p_0 ($p_0 \gg \mu U L/h_0^2$, $p_0 \sim \mu U L/h_0^2$, $\mu U/h_0 \ll p_0 \ll \mu U L/h_0^2$, $p_0 \sim \mu U/h_0$ and $p_0 \ll \mu U/h_0$). In the following, we look at each of these choices, and their effect on Equations (5.14) and (5.15):

1. $p_0 \gg \mu UL/h_0^2$: Equations (5.14) and (5.15) become dominated by the pressure gradient terms; thus to leading order

$$0 = -\frac{h_0^2 p_0}{\mu UL}\frac{\partial p^*}{\partial x^*} \quad \Rightarrow \quad \frac{\partial p^*}{\partial x^*} = 0,$$

$$0 = -\frac{h_0 p_0}{\mu U}\frac{\partial p^*}{\partial y^*} \quad \Rightarrow \quad \frac{\partial p^*}{\partial y^*} = 0,$$

and hence p^* is uniform in space. Since only the pressure gradient comes into play in fluid mechanics, this shows this scaling is the wrong choice, as to leading order there is no pressure gradient.

2. $p_0 \sim \mu UL/h_0^2$: For simplicity let us choose $p_0 = \mu UL/h_0^2$. Then Equations (5.14) and (5.15) respectively become

$$0 = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \quad 0 = -\frac{1}{\varepsilon}\frac{\partial p^*}{\partial y^*} + \varepsilon\frac{\partial^2 v^*}{\partial y^{*2}}. \tag{5.16}$$

The term multiplying ε in the y-equation above can therefore be ignored, leading to

$$0 = -\frac{\partial p^*}{\partial y^*}, \tag{5.17}$$

and therefore p depends on x and t only. This scaling does not lead to a contradiction and is thus a possibility.

3. $p_0 \ll \mu UL/h_0^2$ (we can group the final three choices as we only need to consider Equation (5.14)): Equation (5.14) is now dominated by the viscous term, and thus to leading order

$$0 = \frac{\partial^2 u^*}{\partial y^{*2}}.$$

This has general solution $u^* = c_1(t^*, x^*)y^* + c_2(t^*, x^*)$, where c_1 and c_2 are functions of integration. We then apply no-slip boundary conditions at the side walls $y = y_1$ and $y = y_2$ of the channel. If the side walls are fixed, this will force both c_1 and c_2 to be equal to zero, as there are two zero boundary conditions on the two sides. In turn, this means $u^* = 0$, which is not possible as we assumed that u^* has an order of magnitude 1 when choosing the scale U. In turn, this means that this choice of scaling is incorrect. Even if one or both side walls are moving, the boundary conditions completely determine the values of c_1 and c_2, meaning that the velocity can't have the correct scale.

Thus, we are left with the only consistent choice $p_0 = \mu UL/h_0^2$, and we have the reduced set of equations, derived from (5.13) and (5.16):

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \qquad 0 = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{5.18}$$

where p^* is independent of y^*. Redimensionalising these, we get the lubrication equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad 0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \tag{5.19}$$

with p independent of y.

Step 6: Applying the boundary conditions Since p is a function of x only (and is independent of y), we can integrate Equation (5.19b) with respect to y twice to get

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + Ay + B,$$

where $A(t,x)$ and $B(t,x)$ are functions of integration that are set by the boundary conditions on the two side walls, $u = u_1$ at $y = y_1$ and $u = u_2$ at $y = y_2$:

$$u_1 = \frac{1}{2\mu}\frac{\partial p}{\partial x}y_1^2 + Ay_1 + B, \qquad u_2 = \frac{1}{2\mu}\frac{\partial p}{\partial x}y_2^2 + Ay_2 + B,$$

which can be solved simultaneously to give expressions for A and B:

$$A = \frac{u_2 - u_1}{h} - \frac{(y_1 + y_2)}{2\mu}\frac{\partial p}{\partial x}, \qquad B = \frac{y_2 u_1 - y_1 u_2}{h} + \frac{y_1 y_2}{2\mu}\frac{\partial p}{\partial x},$$

and hence

$$u = \frac{u_1(y_2 - y) + u_2(y - y_1)}{h} - \frac{1}{2\mu}\frac{\partial p}{\partial x}(y_2 - y)(y - y_1). \tag{5.20}$$

Thus u is composed of a linear part that satisfies the boundary conditions (the first term) plus a parabolic part driven by the pressure gradient (the second term).

Step 7: Finding $\partial p/\partial x$ There are two ways to proceed to find $\partial p/\partial x$:

1. If an expression for v is required, we can solve Equation (5.19) for v:

$$\begin{aligned}\frac{\partial v}{\partial y} = \frac{1}{h^2}\frac{\partial h}{\partial x}&(u_1(y_2 - y) + u_2(y - y_1)) \\ &- \frac{1}{h}\left(\frac{\partial u_1}{\partial x}(y_2 - y) + u_1\frac{\partial y_2}{\partial x} + \frac{\partial u_2}{\partial x}(y - y_1) - u_2\frac{\partial y_1}{\partial x}\right) \\ &+ \frac{1}{2\mu}\frac{\partial^2 p}{\partial x^2}(y_2 - y)(y - y_1) + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{\partial y_2}{\partial x}(y - y_1) - \frac{\partial y_1}{\partial x}(y_2 - y)\right),\end{aligned} \tag{5.21}$$

   which can then be integrated (as it is an explicit function of y) to give an expression with one constant of integration. The expressions get too complicated to write down now, though they are typically simpler in the particular case that is to be considered. There are two boundary conditions to be satisfied on v at the side walls: $v = v_1$ at $y = y_1$ and $v = v_2$ at $y = y_2$. One determines the constant of integration, whilst the other boundary condition leads to a second-order ordinary differential equation for p. The red blood cell example below shows how this method works.

2. Alternatively, if we don't require an explicit expression for v, we integrate the continuity equation across the channel:

$$\begin{aligned}0 &= \int_{y_1}^{y_2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\mathrm{d}y \\ &= \frac{1}{2}(u_1 - u_2)\left(\frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x}\right) + \frac{h}{2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}\right) - \frac{h^3}{12\mu}\frac{\partial^2 p}{\partial x^2} - \frac{h^2}{4\mu}\frac{\partial p}{\partial x}\frac{\partial h}{\partial x} + v_2 - v_1,\end{aligned}$$

   and thus we get a second-order differential equation for p (which should be the same as the one obtained by the first method):

$$\frac{\partial^2 p}{\partial x^2} + \frac{3}{h}\frac{\partial h}{\partial x}\frac{\partial p}{\partial x} + \frac{12\mu}{h^3}(v_1 - v_2) + \frac{6\mu}{h^3}(u_2 - u_1)\left(\frac{\partial y_1}{\partial x} + \frac{\partial y_2}{\partial x}\right) - \frac{6\mu}{h^2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}\right) = 0. \tag{5.22}$$

Either way, we end up with a second-order differential equation (5.22) for p, for which we should know all the coefficients from consideration of the boundary conditions. We solve this and apply boundary conditions at the ends of the channel to determine the unknown constants of integration.

Comments

- This method was done in two dimensions, but can be straightforwardly extended to three dimensions.
- In practice in a real problem, start from Equations (5.19) (or similar). These equations would either be given in a question, or you would have to work through guided steps to derive them (see examples in past papers). The algebra can get slightly hairy at times, but the method is otherwise standard.
- When doing a real problem, both the value of ε and that of the reduced Reynolds number must be checked before assuming that the equations of lubrication theory hold. If they are small, the relative error made in making this assumption is the greater of the two values of ε^2 and the reduced Reynolds number.
- If the reduced Reynolds number is slightly larger, that is $\varepsilon^2\mathrm{Re} \ll 1$, but not small enough that the inertial terms can be neglected, then we can improve the accuracy using a series expansion method, as described in Section 5.2, to find the velocity. We return to the governing equations before we removed any terms: (5.13), (5.3) and (5.3). We substitute the pressure scale $p_0 = \mu UL/h_0^2$, and then remove terms that are multiplied by a factor of order ε^2 with respect to the dominant term in each equation. This gives the equations

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{a}$$

$$\varepsilon^2\mathrm{Re}\left(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \tag{b}$$

$$0 = -\frac{\partial p^*}{\partial y^*}. \tag{c}$$

We set

$$\begin{aligned} u^* &= u_0^* + \varepsilon^2\mathrm{Re}\, u_1^* + \left(\varepsilon^2\mathrm{Re}\right)^2 u_2^* + \dots, \\ v^* &= v_0^* + \varepsilon^2\mathrm{Re}\, v_1^* + \left(\varepsilon^2\mathrm{Re}\right)^2 v_2^* + \dots, \\ p^* &= p_0^* + \varepsilon^2\mathrm{Re}\, p_1^* + \left(\varepsilon^2\mathrm{Re}\right)^2 p_2^* + \dots. \end{aligned}$$

Equation (c) gives us that all the p_i^*'s are independent of y^*. Then, solving for u_0^* from Equation (b), v_0^* from (a), u_1^* from (b), v_1^* from (a), *etc* in that order, we obtain the terms in the series.

Example: red blood cell

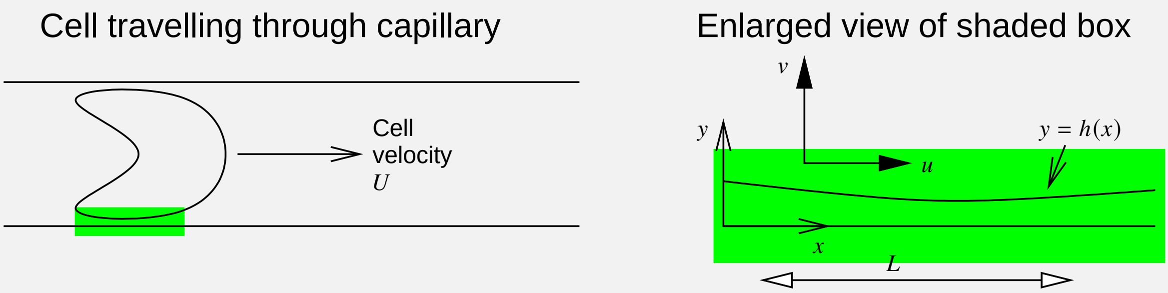


Figure 5.1: Example of a scenario where lubrication theory may be applied. A cell moves steadily with speed U along a vessel with a narrow gap at the walls.

We consider a red blood cell moving along the capillary shown in Figure 5.1 and find the flow and pressure in the blood plasma filling the narrow gap between the cell and the vessel wall. We make the following assumptions:

- The cell travels with constant velocity U parallel to the vessel wall.
- The flow is steady in the frame travelling with the cell (the cell travels as if it were rigid). Usually, this is a good approximation, although, for example it excludes the period just after the cell has entered the capillary.
- The gap is sufficiently narrow that we can model the wall of the vessel as a flat plate[a].

For simplicity we work in Cartesian coordinates, we put the vessel wall at $y = 0$ and the boundary of the cell at $y = h$, so that h is the width of the gap. The boundary conditions are

$$\text{At } y = 0: \quad u = -U, \quad v = 0,$$
$$\text{At } y = h: \quad u = v = 0,$$

(in the notation above, these give us $y_1 = 0$, $y_2 = h$, $u_1 = -U$, $u_2 = v_1 = v_2 = 0$).

To estimate the reduced Reynolds number we use the following approximate parameter values[b]:

Parameter	Symbol	Approx. value
the typical velocity of blood cell	U	1 mm/s
typical gap between cell and wall	h	1 μm
plasma viscosity (approx. viscosity of water)	ν	10^{-6} m^2/s
length of the capillary segment of interest ((approx. cell length)	L	10 μm

With these values, $\varepsilon = 0.1$ and $\varepsilon^2\mathrm{Re} = (h/L)^2(UL/\nu) \approx (0.1)^2 \times (10^{-3} \times 10^{-5}/10^{-6}) = 10^{-4}$, which is a very small value! Therefore we can solve the simplified lubrication equation (5.19) to get (5.20),

$$u = -U\frac{(h-y)}{h} - \frac{1}{2\mu}\frac{\partial p}{\partial x}y\,(h-y)\,.$$

We use Equation (5.19) to get (5.21),

$$\frac{\partial v}{\partial y} = \frac{Uy}{h^2}\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{1}{2\mu}\frac{\mathrm{d}^2 p}{\mathrm{d}x^2}y\,(h-y) + \frac{1}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}h}{\mathrm{d}x}y,$$

and integrating (subject to the boundary condition $v = 0$ at $y = 0$), we find that

$$v = \frac{Uy^2}{2h^2}\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{1}{12\mu}\frac{\mathrm{d}^2 p}{\mathrm{d}x^2}y^2\left(3h - 2y\right) + \frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}h}{\mathrm{d}x}y^2.$$

Enforcing the boundary condition $v = 0$ on $y = h$ leads to the relationship

$$\begin{aligned} 0 = \frac{U}{2}\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{1}{12\mu}h^3\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} + \frac{1}{4\mu}h^2\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{U}{2}\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{1}{12\mu}\left(h^3\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} + 3h^2\frac{\mathrm{d}h}{\mathrm{d}x}\frac{\mathrm{d}p}{\mathrm{d}x}\right) \\ &= \frac{U}{2}\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{1}{12\mu}\frac{\mathrm{d}}{\mathrm{d}x}\left(h^3\frac{\mathrm{d}p}{\mathrm{d}x}\right), \end{aligned}$$

and, integrating the above expression with respect to x gives

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{c - 6\mu hU}{h^3},$$

where c is a constant of integration, which can be solved to find the pressure (note that h needs to be specified to do this). In the case of a flat cell parallel to the wall, h is constant and so $\mathrm{d}p/\mathrm{d}x$ is also constant, and the pressure drops linearly along the vessel, at a rate determined by the conditions at the ends of the cell.

[a] It is not technically difficult to extend this analysis to allow for the curvature of the wall and the non-flat endothelial surface.

[b] Note that most of these parameter values are quoted only as order-of-magnitude estimates because (1) these parameters vary a lot between different vessels and different situations, and (2) only a rough estimate of $\varepsilon^2 Re$ shows that it is well small enough to use the small-reduced-Reynolds-number approximations. The length L is hard to estimate because it depends on the geometry of the vessel chosen. Here we take it as approximately 10 vessel diameters

5.4 Boundary Layer Analysis

⚠ This topic is optional and provided for further reading.

Motivation As its name suggests, boundary layer analysis is used to analyse the flow near a boundary of the fluid. This has important applications in many areas of engineering, because mechanical problems, such as flow instability or separation, often begin in the boundary layer. Thus, analysing the boundary layer flow can lead to a greater understanding of why instabilities or separation develop, which may enable them to be avoided. The key to the analysis is to note that properties vary rapidly as we move away from the boundary, but only moderately along the boundary. In this way, it is similar to lubrication theory, and the derivation is very similar to that in Section 5.3. The main difference is that the thickness of the boundary layer, that is, the analogy of the quantity h that was used in the lubrication theory section, is now not fixed a priori (in the lubrication theory section h was fixed by the geometry).

Another motivation for this field of study is that it is often challenging to simulate boundary layer analysis numerically because the flow properties change very rapidly there. Boundary layer analysis can provide a way either to avoid needing to simulate the boundary layer, or to simplify the equations there so that they are easier to simulate. In addition, boundary layer analysis can be combined with another simplified analysis of the flow away from the boundary to provide a complete solution.

The history of boundary layer analysis can be traced back to the 19th century. Albeit the N-S equation has been formulated early since the mid-1800s, it could not be solved except for the flow in simple geometries (*e.g.*, straight pipe). In 1904, Ludwig Prandtl (1875-1953) first proposed the boundary layer approximation; in his idea, the flow is divided into 2 regions (Figure 5.2):

- **outer flow region**: flow can be approximated as *inviscid* and *irrotational*; the velocity field in this region is solvable using the continuity equation and Euler equation (simplified from N-S equation for inviscid fluid flow), and the pressure field is solved using Bernoulli's theorem.
- **inner flow region**: flow near the wall, where viscous effects and rotationality cannot be neglected. We need to solve the boundary layer equation.

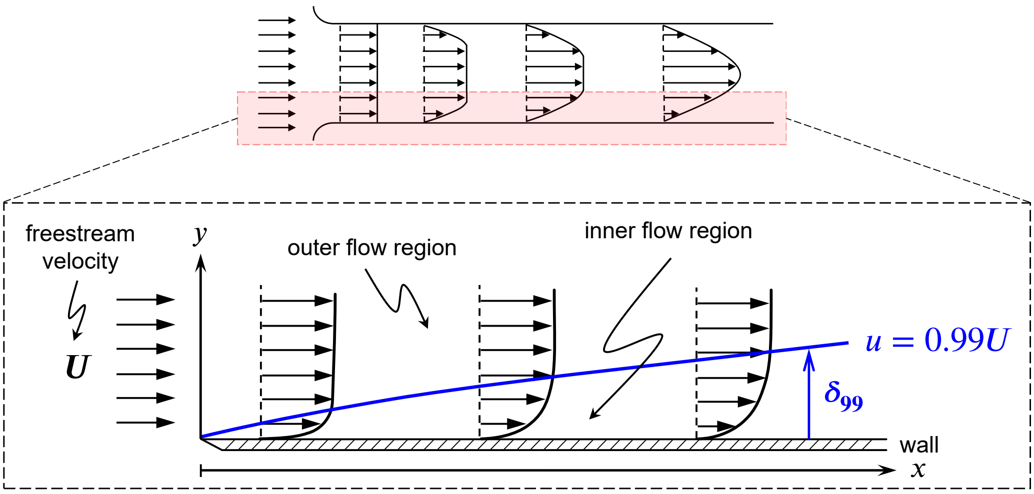


Figure 5.2: A flat plate parallel to an oncoming flow. The near wall region is where the boundary layer exists, where the viscous effects dominate and influence the flow. δ_{99} denotes the boundary layer thickness at which $u = 99\%U$ (*i.e.*, 99% recovery of the free-stream velocity). Note that δ_{99} is NOT a streamline!

For simplicity, we will assume a two-dimensional, steady flow near a flat boundary parallel to the x-axis, although the theory can be straightforwardly extended to a three-dimensional, unsteady flow near a non-flat surface. In this section, we will derive the width of the boundary layer and simplified equations for the flow within it.

The velocity and pressure fields are governed by the equations

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\mathrm{d}p}{\mathrm{d}x}+\mu\frac{\partial^2 u}{\partial y^2}, \tag{5.23}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{5.24}$$

where the pressure p is a function of x only. Equations (5.23) and (5.24) represent a considerable simplification of the full Navier-Stokes and continuity equations.

Boundary Layer Equation The boundary layer equation is an approximation to the N-S equation. To derive such, we need to nondimensionalise the x-component of the N-S momentum equation. Starting by defining the nondimensional variables

$$x^*=\frac{x}{L},\qquad y^*=\frac{y}{\delta},\qquad u^*=\frac{u}{U},\qquad v^*=\frac{v}{V},\qquad p^*=\frac{p}{P_0}=\frac{p}{\rho U^2}$$

where L is the characteristic length scale, δ is the thickness of the boundary layer, U, V are the velocity scales in the x- and y-directions, respectively. $P_0=\rho U^2$ is the characteristic pressure, derived from Bernoulli's theorem.

---

1. The nondimensional continuity equation is

$$\frac{U}{L}\frac{\partial u^*}{\partial x^*}+\frac{V}{\delta}\frac{\partial u^*}{\partial y^*}=0. \tag{5.25}$$

   Note that, to satisfy the nondimensional continuity equation, the order of magnitude of the first term must be balanced to that of the second term, *i.e.*, $\frac{U}{L}$ and $\frac{V}{\delta}$ should be of the same order of magnitude:

$$O\left(\frac{U}{L}\right)+O\left(\frac{V}{\delta}\right)=0,\quad\Rightarrow\quad\frac{U}{L}\sim\frac{V}{\delta}\quad\Rightarrow\quad V\sim\frac{U\delta}{L} \tag{5.26}$$

2. The nondimensional x-momentum equation is

$$\frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*}+\frac{UV}{\delta}v^*\frac{\partial u^*}{\partial y^*}=-\frac{U^2}{L}\frac{\partial p^*}{\partial x^*}+\nu\frac{U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right). \tag{5.27}$$

   To further simplify this equation, we can take a few actions

   - Use the relation derived from Equation 5.26 to eliminate V from Equation 5.27, *i.e.*, $\frac{UV}{\delta}=\frac{U}{\delta}\cdot\frac{U\delta}{L}=\frac{U^2}{L}$;
   - Multiply Equation 5.27 by the term L/U^2.

   So far, the nondimensional x-momentum equation looks like

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right). \tag{5.28}$$

   Further,

- We restrict the analysis to 'narrow' channels only: $L/\delta \gg 1$.
- We are interested in the type of flow that $\mathrm{Re} \gg 1$. This ensures that the $1/\mathrm{Re}$ term is safe to be eliminated.

So far, the revised nondimensional x-momentum equation looks like

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2} \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{5.29}$$

The last question regards the term $\frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2}$, since $1/\mathrm{Re} \ll 1$ but $L/\delta \gg 1$, which term dominates? We know the order of magnitude of the L.H.S. and the R.H.S. of Equation 5.29 must balance:

$$\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2}\right),$$

Obviously, $\mathcal{O}\left(\frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2}\right) = \mathcal{O}(1)$. This means, $\frac{\delta}{L} \sim \mathrm{Re}^{-1/2} \Rightarrow \boxed{\delta \sim \sqrt{L\nu/U}}$.

3. Similarly, the nondimensional y-momentum equation can be simplified as

$$\frac{\partial p^*}{\partial y^*} = 0. \tag{5.30}$$

Re-dimentionalise Equation 5.25, Equation 5.29, and Equation 5.30, which are the boundary layer equations:

$$\text{(mass)} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5.31}$$

$$(x\text{-momentum}) \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{5.32}$$

$$(y\text{-momentum}) \quad \frac{\partial p}{\partial y} = 0. \tag{5.33}$$

Comments

- The actual width of the boundary layer is not precisely defined; $\delta \sim \sqrt{L\nu/U}$ is only an order-of-magnitude estimate. The point is that the flows well outside and well within the boundary layer are qualitatively different from one another since different physical effects play a dominant role.
- Boundary layer analysis is a huge topic in its own right, and we have only scratched the surface here! For example, we could generalise this approach to include the following:
  - dependence upon the third spatial dimension,
  - time-dependence of the solution,
  - gravity,
  - turbulence,
  - multi-layer boundary layers, in which different effects become important at different distances from the surface,

- ....

Boundary Conditions For the type of the flow as illustrated in Figure 5.2, the boundary conditions are

$$\begin{aligned} u = U, &\quad \text{at } x = y = 0 \\ u = v = 0, &\quad \text{at } y = 0, x \neq 0 \\ u = U, &\quad \text{as } y \to \infty \end{aligned}$$

Displacement Thickness The boundary layer thickness, δ_{99} can be difficult to measure directly. One alternative approach is finding the equivalence of δ_{99} with the displacement thickness, $\delta_1(x)$. As illustrated by Figure 5.3(a), $\delta_1(x)$ is a thin plate that *obstructs* the inviscid flow (stagnant layer).

The expression of $\delta_1(x)$ is derived by mass conservation: equating the total mass flow at the inlet and at the inviscid (unobstructed) region,

$$\rho \int_0^\infty u(x, y)\mathrm{d}y = \rho \int_{\delta_1}^\infty U\mathrm{d}y.$$

Divide both sides by ρU, then split the integral,

$$\rho \int_0^\infty u^*\mathrm{d}y = \int_{\delta_1}^\infty \mathrm{d}y \implies \int_0^\infty u^*\mathrm{d}y = \int_0^\infty \mathrm{d}y - \int_0^{\delta_1} \mathrm{d}y \implies \boxed{\delta_1(x) = \int_0^\infty (1 - u^*)\mathrm{d}y}.$$

Momentum Thickness The momentum thickness, $\delta_2(x)$, is an alternative approximation of the boundary layer thickness, for which $\delta_2(x)$ has the same momentum deficit as the actual boundary layer profile, as shown by Figure 5.3(b).

Equating the 'artificial' momentum deficit created by δ_2 to the real momentum deficit raised from the velocity deficit, we have

$$\underbrace{\rho \int_0^{\delta_2} U^2\,\mathrm{d}y}_{\text{momentum deficit by } \delta_2} = \int_0^\infty \rho u \cdot \underbrace{(U - u)}_{\text{velocity deficit}} \,\mathrm{d}y \implies \boxed{\delta_2(x) = \int_0^\infty u^*(1 - u^*)\,\mathrm{d}y}.$$

Despite the abstraction that lies in the concept of momentum thickness, it is particularly useful in finding the fluid drag and skin friction on the plate.

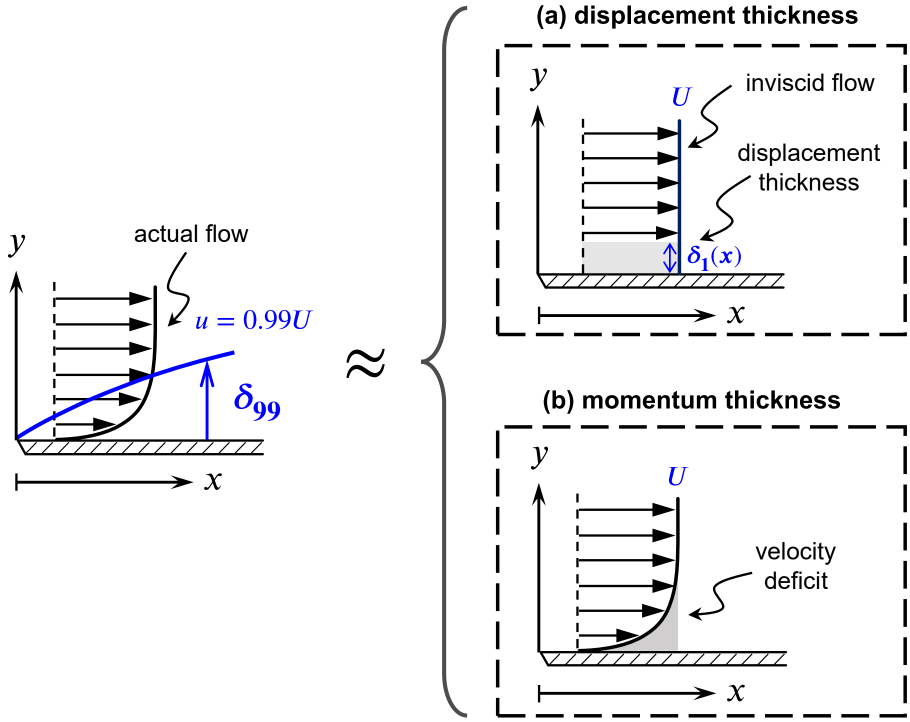


Figure 5.3: Two approximations of the thickness of an actual boundary layer: (a) displacement thickness and (b) momentum thickness.

Example: Aorta

In the aorta we have $U \sim 1$ m/s, $L \sim 0.5$ m (length of torso), $\mu \approx 0.004$ Pa s and $\rho \approx 1,000$ kg/m^3. Thus $\mathrm{Re} = \dfrac{\rho U L}{\mu} = \dfrac{1000 \times 1 \times 0.5}{0.004} \approx 10^5$, which is the Reynolds number based on the length. We have a boundary layer of thickness $\sqrt{\dfrac{L\mu}{U\rho}} = \sqrt{\dfrac{0.5 \times 0.004}{1 \times 1,000}} \approx 10^{-3}$ m $= 1$ mm. Thus the boundary layer is about 1/40 of the diameter (1/20 of the radius) of the vessel. If we did a numerical simulation of this, we would require several points within the boundary layer (because we need to resolve on a scale much smaller than the width of the boundary layer). Thus, we would need to use a mesh spacing of much less than 1/40 of the diameter.

We have previously estimated Reynolds number based on the diameter for vessels, which - in the case of the aorta (with diameter 4 cm) - would give $\mathrm{Re} \approx 10^4$! This is a good example that illustrates the reason why it is important to specify on which dimension we are basing our estimate of the Reynolds number (if it is not obvious).

5.5 Flow Passing Around a Sphere

⚠ This topic is optional and provided for further reading.

From subsection 5.2, we have derived the Stokes equation, as

$$\mu \nabla^2 \mathbf{u} = \nabla p.$$

The type of flow governed by the Stokes equation is known as the Stokes flow, or the creeping flow. Possibly, the most famous example of the Stokes flow is the flow passing around a sphere, as shown in Figure 5.4. In Bioengineering, Stokes flow could, for example, be the drag experienced by a near-spherical swimming microorganism.

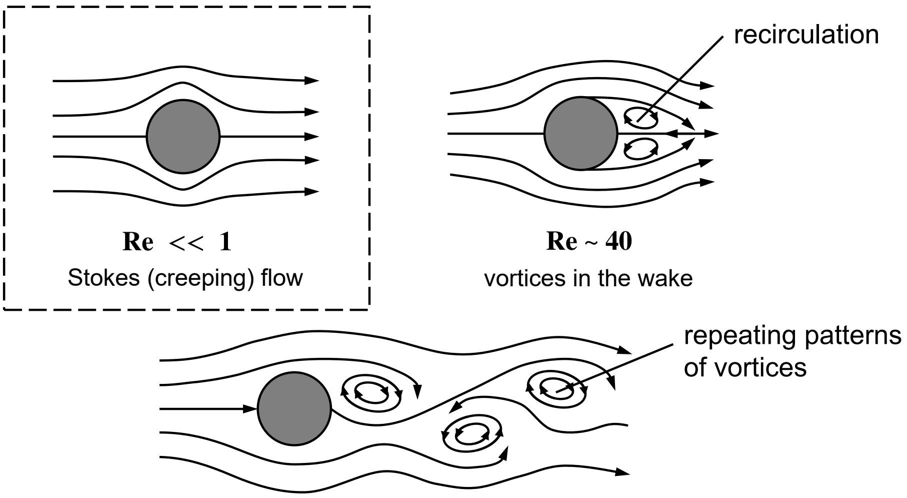


Figure 5.4: Flow passing around a circular obstacle at different Reynolds numbers. The top left scenario depicts the Stokes flow when $\text{Re} \ll 1$ - note that there is no flow separation or vortices.

The analytical solution for the Stokes flow exists, albeit the derivation is cumbersome, which involves the use of the streamfunction. The analytical result shows that the drag force experienced by an object moving through a fluid at low Reynolds numbers is called the Stokes drag, and is derived from the Stokes equations. On a sphere, the Stokes drag is

$$F_d = 6\pi\mu U a \tag{5.34}$$

in the direction opposing the motion, which holds as long as $\rho U a/\mu \ll 1$. This is a famous classical result in fluid mechanics.

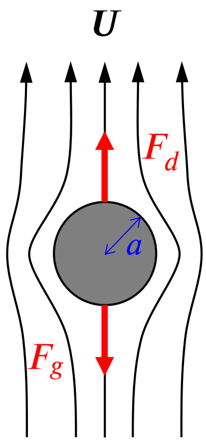


Figure 5.5: Schematic of the Stokes flow. The Stokes drug, F_d, is balanced by the force of gravity, F_g.

For example, the principle is used in a falling ball viscometer in which a small spherical ball of known radius a and mass m is dropped into a fluid, whose viscosity μ is required to be measured.

Once the ball has reached its terminal falling velocity U, the forces on it must be in equilibrium, meaning that the force of gravity balances the drag force (Figure 5.5). Assuming the sphere is small enough and going slowly enough, and the fluid is viscous enough that the flow has a low Reynolds number, the drag is given by $6\pi\mu Ua$ and the force of gravity is mg, where g is the acceleration due to gravity. Thus

$$\mu = \frac{mg}{6\pi Ua}, \tag{5.35}$$

and so, measuring the terminal velocity and substituting it into this equation gives an estimate of the viscosity of the fluid.

Example

Although Stokes flow occurs when $\mathrm{Re} \ll 1$, it is also worth noting that, at higher Re, additional unsteady forces appear. As illustrated in the bottom sketch in Figure 5.4, the repeating patterns of vortices in the wake region of the obstacle contribute to an effect known as vortex shedding, which can generate oscillatory forces on the body and lead to flow-induced vibrations.

As such, in structural dynamics, damping mechanisms are often introduced to dissipate energy and reduce vibration amplitudes. A well-known example is the tuned mass damper (TMD) installed in the Taipei 101. As shown in Figure 5.6, a large auxiliary mass is suspended within the structure and tuned to dominant vibration frequency of the building. When wind-induced loads excite the tower, the TMD oscillates out of phase with the structural motion, absorbing vibrational energy and thereby reducing the overall response of the structure.

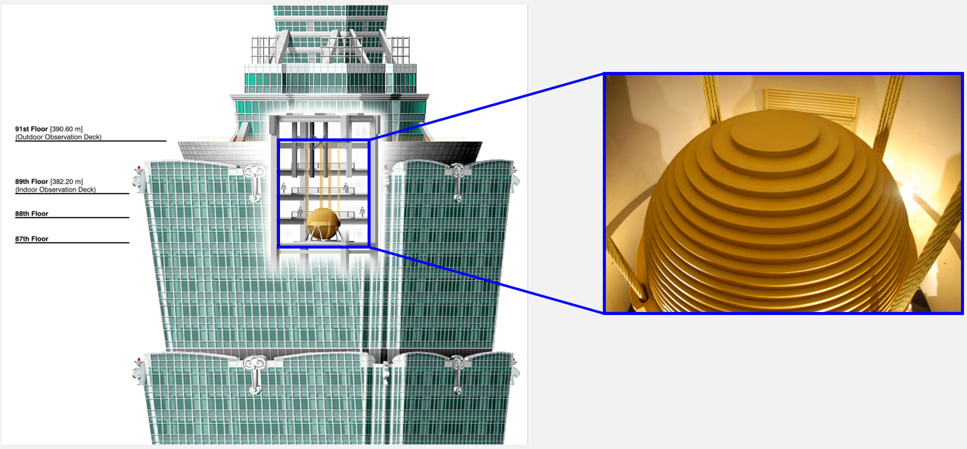


Figure 5.6: The tuned mass damper in Taipei 101 Tower. (Wikipedia)

6 Physiological Modelling

6.1 Lumped Parameter Modelling

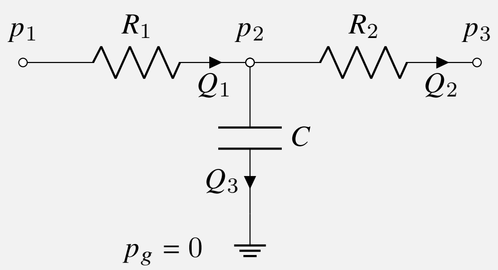
Resistance, Compliance, and Inertance

Resistance	Compliance	Inertance
$Q = \Delta p/R$	$Q = C\dfrac{\partial p}{\partial t}$	$p = L\dfrac{\partial Q}{\partial t}$

- Resistance R: analogous to the electrical resistance, which models the dissipation of energy. The flow rate Q is analogous to the electrical current (usually denoted by I), and the pressure p is analogous to the electrical voltage (usually denoted by V).
- Compliance C: analogous to the electrical capacitor, which models the expansion of cardiovascular chambers under pressure, allowing them to store more fluid.
- Inertance L: analogous to the electrical inductor, which models the inertial effects of the fluid. When the fluid momentum is substantial, as the pressure on forward-flowing fluid reverses, the fluid will not suddenly reverse its direction, but decelerate over a transient.

Example: Solving a Lumped Parameter Network

Consider the example lumped parameter network shown below,



... which yields a linear system with 4 unknowns (p_2, Q_1, Q_2, Q_3) and 4 simultaneous equations:

$$\begin{cases} p_2 - p_1 = R_1 Q_1, \\ p_3 - p_2 = R_2 Q_2, \\ Q_3 = C(p_2^{(t)} - p_2^{(t-1)})/\Delta t, \\ Q_1 = Q_2 + Q_3. \end{cases}$$

Note that $p_2^{(t-1)}$ denotes the pressure p_2 at the previous time step $t-1$; $(p_2^{(t)} - p_2^{(t-1)})/\Delta t$ is an approximation of the derivative of p w.r.t. t in the backward Euler fashion.

The above linear system can be arranged into a matrix system, $\mathbf{Ax} = \mathbf{b}$,

$$\begin{bmatrix} 1 & -R_1 & 0 & 0 \\ -1 & 0 & -R_2 & 0 \\ -1 & 0 & 0 & \frac{\Delta t}{C} \\ 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p_2 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ -p_3 \\ -p_2^{(t-1)} \\ 0 \end{bmatrix},$$

and can be easily solved by inversion of the coefficient matrix: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

6.2 Windkessel Models

The Windkessel models are a category of lumped parameter models used to mathematically describe the blood pressure waveform in the large, elastic arteries.

Historically, the term "windkessel" refers to the air chamber used in early German fire engines, which temporarily stores energy by compressing air when fluid is pumped in and then releases that energy to maintain a more continuous flow (Figure 6.1). By analogy, the arterial Windkessel represents the ability of large arteries to store blood during systole through elastic expansion and to release it during diastole, thereby smoothing the pulsatile output of the heart into a more continuous peripheral flow.

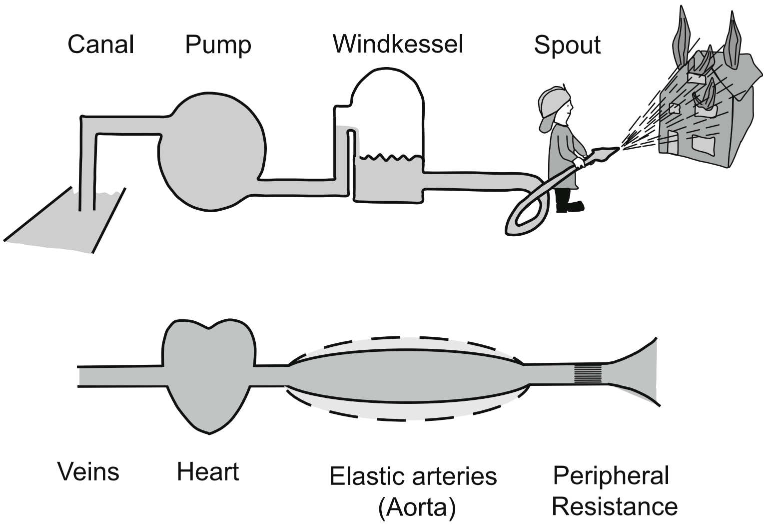
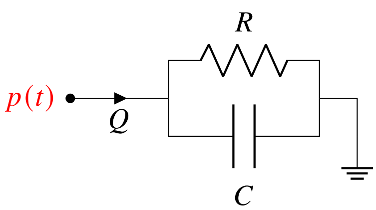


Figure 6.1: The concept of the Windkessel. The air reservoir is the actual Windkessel, and the large arteries act as the Windkessel. (Westerhof *et. al.*, 2009)

⚠ Kindly note that in the following notes, Z_c and R are used to denote proximal (characteristic) resistance and distal resistance in the Windkessel models; while in the lecture slides (and some other materials), they are denoted as R_1 and R_2. Conceptually and mathematically, $Z_c \leftrightarrow R_1$ and $R \leftrightarrow R_2$ are equivalent.

Two-element Windkessel Model

This is the "original" Windkessel model proposed by Otto Frank (1865-1944).



Governing Equation:

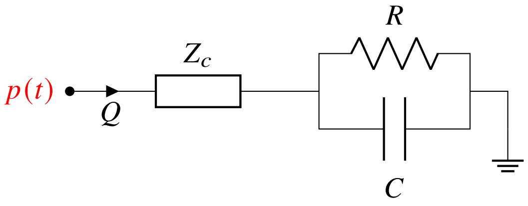
$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{p(t)}{RC} = \frac{Q}{C}$$

where C denotes the vessel compliance (elasticity), R denotes the peripheral (distal) resistance.

Three-element Windkessel Model

One ostensible limitation of the two-element Windkessel model is that it cannot accurately predict the upstroke pressure waveform in early systole (*i.e.*, the rise in pressure during early systole), but simply the diastolic pressure as a monotonic exponential decay (*i.e.*, $p(t) = p_{\text{init}} \cdot e^{-t/RC}$).

In the three-element Windkessel model, a characteristic impedance component, Z_c, is introduced to represent the impedance of the proximal large arteries. This element accounts for the instantaneous pressure-flow relationship associated with wave propagation during early systole, thereby enabling a more realistic representation of the systolic pressure upstroke.



Governing Equation:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{p(t)}{RC} = \frac{Q}{C}\left(1 + \frac{Z_c}{R}\right) + Z_c\frac{\mathrm{d}Q}{\mathrm{d}t}$$

where Z_c is the characteristic impedance.

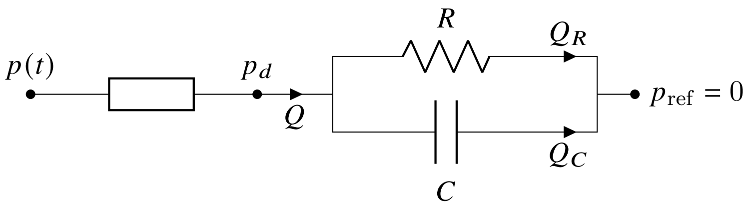
Comments

Rigorously, the term “impedance” governs how the pressure responds to the pulsatile flow; whereas the term “resistance” characterises viscous energy loss under steady (or time-averaged) flow conditions. Therefore, impedance generalises the steady pressure-flow relation to unsteady flow and is frequency-dependent.

This is indeed the interpretation of the characteristic impedance, *i.e.*, the unsteady, high-frequency pressure-flow ratio set by proximal arterial stiffness and wave speed (which we shall discuss in Section 6.3).

Derivation

For the full derivation of the three-element Windkessel model, consider the electrical schematic annotated below: Q is the total flow, p_d is the distal pressure defined at the junction of the RC network.



Apply Kirchhoff’s Current Law at node p_d: $Q = Q_R + Q_C$. Moreover, since $p(t) - p_d = Z_cQ \implies p_d = p(t) - Z_cQ$.

- The flow passes through the distal resistance R is Q_R:

$$Q_R = \frac{p_d}{R} = \frac{p(t) - Z_cQ}{R} = \frac{p(t)}{R} - \frac{Z_cQ}{R}.$$

- the flow passes through the capacitor C is Q_C:

$$Q_C = C\frac{\mathrm{d}p_d}{\mathrm{d}t} = C\frac{\mathrm{d}[p(t) - Z_cQ]}{\mathrm{d}t} = C\frac{\mathrm{d}p(t)}{\mathrm{d}t} - CZ_c\frac{\mathrm{d}Q}{\mathrm{d}t}.$$

Hence, the total flow Q is

$$\begin{aligned} Q &= Q_R + Q_C \\ &= \frac{p(t)}{R} - \frac{Z_c Q}{R} + C\frac{\mathrm{d}p(t)}{\mathrm{d}t} - CZ_c\frac{\mathrm{d}Q}{\mathrm{d}t}, \end{aligned}$$

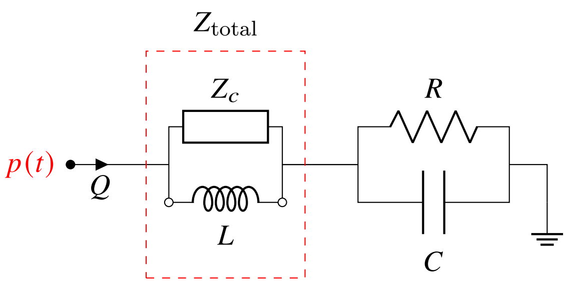
rearrange, we get

$$C\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{p(t)}{R} = \left(1 + \frac{Z_c}{R}\right)Q + CZ_c\frac{\mathrm{d}Q}{\mathrm{d}t}.$$

Divide both sides of the equation above by C, and we will get the final governing equation as presented.

Four-element Windkessel Model

The four-element Windkessel model is a further expansion of the three-element Windkessel model. An inductor component, L, is introduced, in parallel to the characteristic impedance, to account for the frequency-dependent relationship between pressure and flow, thereby representing blood inertance and capturing variations in arterial pressure response with changes in heart rate (*i.e.*, frequency).



Governing Equation:

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{p(t)}{RC} = \frac{Q}{C}\left(1 + \frac{Z_{\text{total}}}{R}\right) + Z_{\text{total}}\frac{\mathrm{d}Q}{\mathrm{d}t}$$

where $Z_{\text{total}} = \dfrac{\mathrm{i}\omega L Z_c}{\mathrm{i}\omega L + Z_c}$ is the total impedance of the parallel network - the characteristic impedance, Z_c and the inductor, L.

Derivation

Apply Kirchhoff's Current Law at node p_d: $Q = Q_R + Q_C$. However, we need to express p_d in terms of $p(t)$, hence need to solve the total impedance of the Z_c-L parallel network:

$$\frac{1}{Z_{\text{total}}} = \frac{1}{Z_c} + \frac{1}{\mathrm{i}2\pi f L} = \frac{\mathrm{i}2\pi f L + Z_c}{\mathrm{i}2\pi f L Z_c} \quad \Longrightarrow \quad Z_{\text{total}} = \frac{\mathrm{i}2\pi f L Z_c}{\mathrm{i}2\pi f L + Z_c}.$$

Note that sometimes $2\pi f$ is denoted as ω, which is the angular frequency. Now, $p(t) - p_d = Z_{\text{total}}Q$. The rest of this derivation follows the same procedure for 3-WK.

What is the necessity of the inductance? The inclusion of the inductor better captures the frequency characteristics of the flow.

- At the low f range: $2\pi f L \ll Z_c$, hence $Z_{\text{total}} \to 0$, which removes the characteristic impedance in the whole circuit;
- At the high f range: $2\pi f L \gg Z_c$, hence $Z_{\text{total}} \to Z_c$.

This means the inductance has no effect when the flow is steady, providing a zero resistance pathway to the rest of the circuit under steady flow conditions. This effect is shown in Figure 6.2.

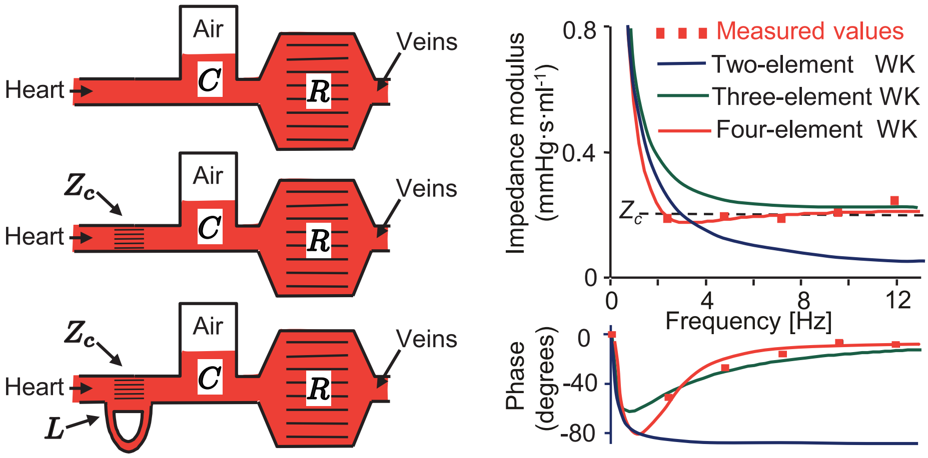


Figure 6.2: Left: the mechanical equivalence of two-element, three-element, and four-element Windkessel models; Right: Comparisons between the clinically measured and modelled (using the three Windkessel models) input impedance against the frequency variations. (Westerhof *et al.*)

6.3 Moens-Korteweg Model of Pulse Wave Velocity

Pulse waves Blood is ejected into the aorta by contraction of the left ventricle, generating a rapidly propagating wave of pressure (not flow) accompanied by deformation of the aortic wall. This wave, also known as the **pulse wave**, travels along the arteries with a much faster speed (by orders of magnitude) than the bulk motion of blood, and undergoes reflections at sites of impedance mismatch such as arterial bifurcations and tapering.

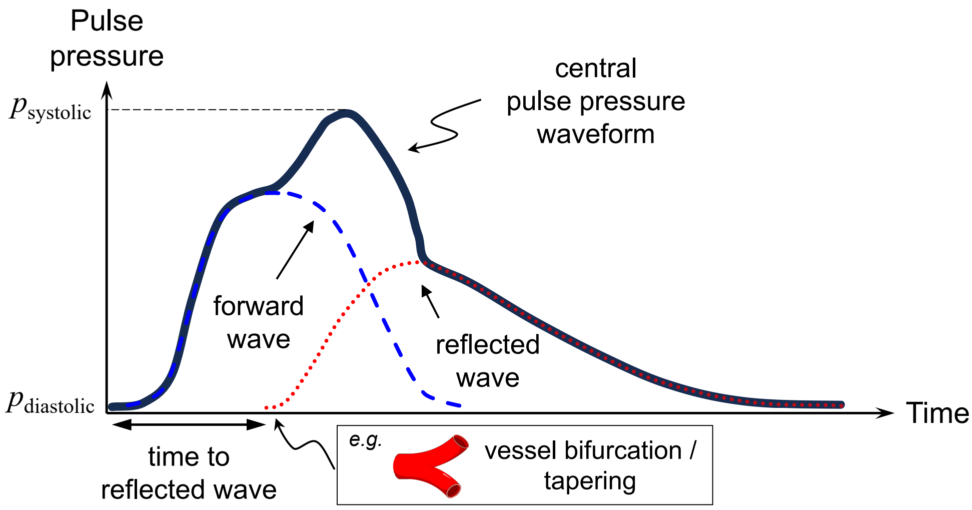


Figure 6.3: Schematic of the formation of the central arterial pressure waveform. The total arterial pressure results from the superposition of the forward-travelling wave (blue) generated by ventricular ejection and the reflected wave (red) arising from impedance mismatches in the arterial tree.

How fast does the pulse wave travel? Adriaan Isebree Moens (1846-1891) and Diederik Korteweg (1848-1941) derived an expression of the pulse wave velocity, PWV, linked to the distensibility of the aortic wall:

$$\mathrm{PWV} = \sqrt{\frac{Eh}{2R\rho}}, \tag{6.1}$$

where E denotes the linear elasticity of the aortic wall, h and R are the lumen thickness and radius, respectively, with $h \ll R$; and ρ is the density of the blood. The Moens-Korteweg model assumes the blood to be inviscid.

By definition, PWV increases with the stiffness of the vessels and decreases with the vessel radius.

Derivation

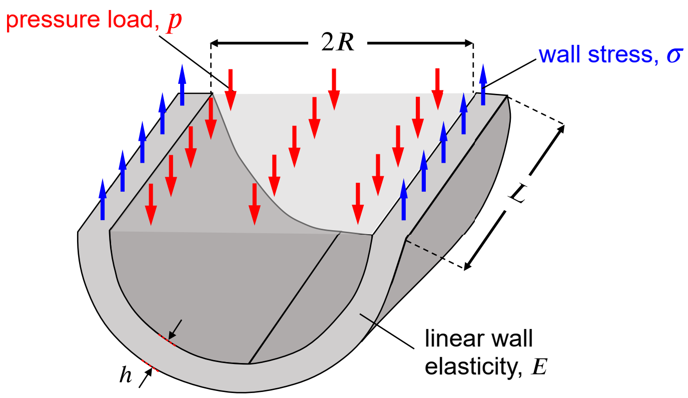


Figure 6.4: The schematic for the derivation of the Moens-Korteweg equation.

Equation 1 Assume the arterial wall has an isotropic linear elasticity (constant Young's modulus, E). Therefore, the stress(σ)-strain(ε) relation is

$$\sigma = E\varepsilon = E\frac{\Delta R}{R} \quad \text{with} \quad \varepsilon = \frac{(2\pi(R+\Delta R) - 2\pi R)}{2\pi R} = \frac{\Delta R}{R}.$$

Applying Newton's 2$^{\text{nd}}$ Law and rearranging the expression leads to an expression of the pressure,

$$m_{\text{wall}} a_{\text{wall}} = F_{\text{pressure}} - F_{\text{wall}}$$
$$0 = 2RL \times p - 2Lh \times \sigma \quad \Rightarrow \quad p = \frac{\sigma h}{R} = \frac{Eh}{R^2}\Delta R.$$

Differentiating p w.r.t. t, this leads to equation 1,

$$\boxed{\frac{\partial p}{\partial t} = \frac{Eh}{R^2}\frac{\partial \Delta R}{\partial t}}.$$

Equation 2 Integrating the continuity equation over the vascular cross-sectional area

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \underbrace{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}_{0,\ \text{axis-symmetrical}} + \frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad \int \left(\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right)\partial A = 0$$
$$\Rightarrow \quad \int_{r=0}^{r=R} \left(\frac{1}{r}\frac{\partial r u_r}{\partial r}\right) 2\pi r \partial r + \pi R^2 \frac{\partial \bar{u}_z}{\partial z} = 0$$
$$\Rightarrow \quad 2\pi R u_R + \pi R^2 \frac{\partial \bar{u}_z}{\partial z} = 0.$$

Re-arrange leads to the equation 2,

$$\boxed{u_r = -\frac{R}{2}\frac{\partial \bar{u}_z}{\partial z}},$$

where the notation $\bar{u}_z$ denotes the average z-velocity across cross-section.

Equation 3 Assume negligible convective acceleration and no viscous losses, the Navier-Stokes z-momentum equation can be simplified as,

$$\rho\left(\frac{\partial u_z}{\partial t} + \cancel{u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}}^{0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}}^{0}\right] + \cancel{\rho f_z}^{0}$$

$$\Rightarrow \boxed{\rho\frac{\partial \overline{u}_z}{\partial t} = -\frac{\partial p}{\partial z}}.$$

Derivation of PWV First, let $u_r = \dfrac{\partial \Delta R}{\partial t}$, this equates equation 1 and equation 2 and leads to equation 4

$$\underbrace{u_r = -\frac{R}{2}\frac{\partial \overline{u}_z}{\partial z}}_{\text{equation 2}} = \underbrace{\frac{\partial \Delta R}{\partial t} = \frac{R^2}{Eh}\frac{\partial p}{\partial t}}_{\text{equation 1}}, \quad \Rightarrow \quad \underbrace{\frac{\partial \overline{u}_z}{\partial z} = -\frac{2R}{Eh}\frac{\partial p}{\partial t}}_{\text{equation 4}}$$

Next, differentiate equation 3 and equation 4 w.r.t. t,

$$\rho\frac{\partial \overline{u}_z}{\partial t} = -\frac{\partial p}{\partial z} \xrightarrow[\text{w.r.t. } t]{\text{differentiate}} \rho\frac{\partial^2 \overline{u}_z}{\partial t \partial z} = -\frac{\partial^2 p}{\partial z^2},$$

$$\frac{\partial \overline{u}_z}{\partial z} = -\frac{2R}{Eh}\frac{\partial p}{\partial t} \xrightarrow[\text{w.r.t. } t]{\text{differentiate}} \frac{\partial^2 \overline{u}_z}{\partial z \partial t} = -\frac{2R}{Eh}\frac{\partial^2 p}{\partial t^2},$$

which allows us to equate the R.H.S. as

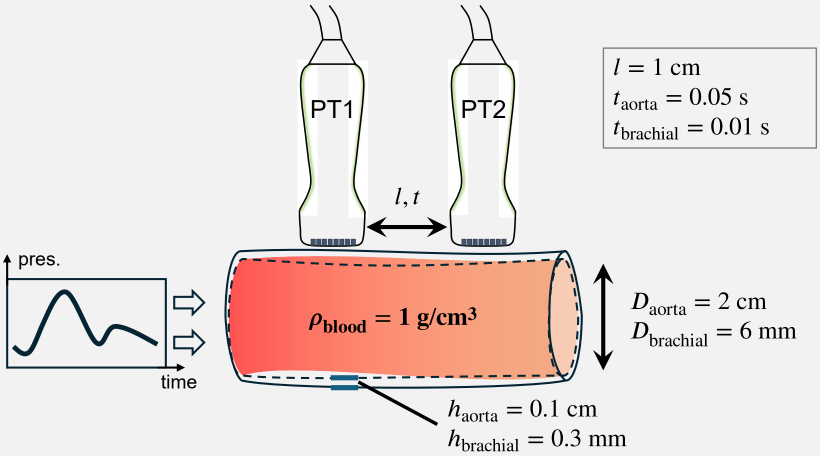
$$\frac{\partial^2 p}{\partial z^2} = \frac{2R\rho}{Eh}\frac{\partial^2 p}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 p}{\partial t^2} = \underbrace{\frac{Eh}{2R\rho}}_{c^2}\frac{\partial^2 p}{\partial z^2},$$

which can be subsequently rearranged as the wave equation. Denote the term $\dfrac{Eh}{2R\rho} = c^2$, for which the term c is the expression of the wave speed of pressure (*a.k.a.* pulse wave velocity, PWV).

Example: Moens-Korteweg equation

Pressure transducers spaced $1\ \mathrm{cm}$ apart axially are deployed in the aorta, and then in the brachial artery of a healthy human. Based on the arrival times of the peaks of the pressure waves, the time delay between signals from the two transducers is measured as $0.05\ \mathrm{s}$ for the aorta and $0.01\ \mathrm{s}$ for the brachial artery.

For the aorta, diameter $= 2\ \mathrm{cm}$, wall thickness $= 0.1\ \mathrm{cm}$. For the brachial artery, diameter $= 6\ \mathrm{mm}$ and wall thickness $= 0.3\ \mathrm{mm}$. You may assume a blood density of $1\ \mathrm{g/cm^3}$.



Question:

- Estimate the elastic moduli of these two vessels;
- Explain differences in their values based on your knowledge of artery wall structure.

Answer: From the Moens-Korteweg equation,

$$c = \sqrt{\frac{Eh}{2\rho R}} \quad \Rightarrow \quad E = \frac{2\rho R c^2}{h},$$

where c is wavespeed, E is the elastic modulus, R is radius, h is wall thickness.

- Aorta: The measured wavespeed is

$$c_{\text{aorta}} = \frac{\ell}{t_{\text{aorta}}} = \frac{1\,\text{cm}}{0.05\,\text{s}} = 20\,\text{cm/s},$$

and thus

$$E_{\text{aorta}} = \frac{2 \times (1\,\text{g/cm}^3) \times (1\,\text{cm}) \times (20\,\text{cm/s})^2}{0.1\,\text{cm}} = 8,000\,\frac{\text{g}}{\text{cm}\,\text{s}^2}.$$

- Brachial artery: The measured wavespeed is

$$c_{\text{brachial}} = \frac{\ell}{t_{\text{brachial}}} = \frac{1\,\text{cm}}{0.01\,\text{s}} = 100\,\text{cm/s},$$

and thus

$$E_{\text{brachial}} = \frac{2 \times (1\,\text{g/cm}^3) \times (0.3\,\text{cm}) \times (100\,\text{cm/s})^2}{0.03\,\text{cm}} = 200,000\,\frac{\text{g}}{\text{cm} \cdot \text{s}^2}.$$

⇒ The brachial is stiffer (higher elastic modulus) than the aorta, due to its higher collagen and smooth muscle cell content, whereas the aorta has more elastin.

Appendix A Derivations of Continuity and Navier-Stokes Equations From Reynolds Transport Theorem

A.1 Reynolds Transport Theorem

To elucidate the concept of the Reynolds Transport Theorem (RTT), we consider a control volume (CV) initially filled with a quantity B that flows at a fixed speed $\mathbf{u}$. After some time, portions of B initially inside the volume move outside and new portions of B enter, as depicted by Figure A.1.

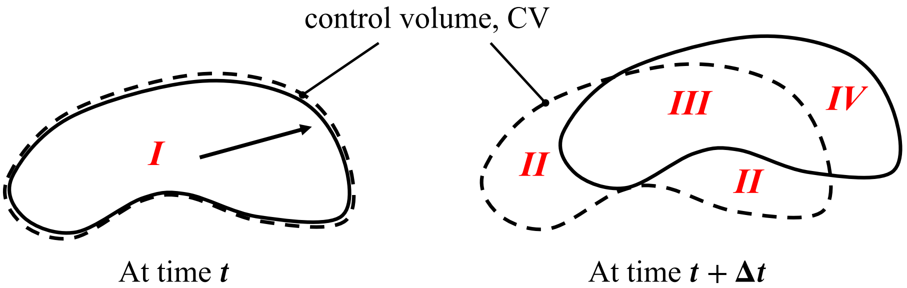


Figure A.1: Movement of a physical quantity B by fluid flow from inside to outside of the control volume.

Regions in Figure A.1 are

- **I**: the entire fluid system within CV at time t
- **II**: new fluid that has entered CV at time $t + \Delta t$
- **III**: portion of fluid system that remains inside CV at time $t + \Delta t$
- **IV**: portion of a fluid system that is outside of CV at time $t + \Delta t$

By conservation of the quantity B in the CV, *"how much out must be balanced by how much in"*,

$$\underbrace{B_{\text{system}}|_{t+\Delta t} - B_{\text{system}}|_t}_{\text{change of B in system}} = B_{III} + B_{IV} - B_I$$

$$= \underbrace{(B_{III} + B_{II} - B_I)}_{\text{change of B in CV}} + \underbrace{(B_{IV} - B_{II})}_{\text{Net amount of B leaving CV due to flow}}$$

$$\underbrace{B_{\text{system}}|_{t+\Delta t} - B_{\text{system}}}_{\textbf{Term A}} = \underbrace{B_{CV}|_{t+\Delta t} - B_{CV}|_t}_{\textbf{Term B}} + \underbrace{\text{Net amount of B leaving CV due to flow}}_{\textbf{Term C}}$$

Divide each term by Δt, and limit the change in time to infinitesimally small: $\Delta t \to 0$.

Term A: rate of change of B within the system (Lagrangian description)

$$\lim_{\Delta t \to 0} \frac{B_{\text{system}}|_{t+\Delta t} - B_{\text{system}}|_t}{\Delta t} = \frac{\mathrm{d}B_{\text{system}}}{\mathrm{d}t}.$$

Term B: rate of change of B within CV (Eulerian description)

$$\lim_{\Delta t \to 0} \frac{B_{CV}|_{t+\Delta t} - B_{CV}|_t}{\Delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t} \int_{\text{CV}} \rho\beta \, \mathrm{d}V,$$

where $\beta = \dfrac{\mathrm{d}B}{\mathrm{d}m}$ is the amount of B per unit mass.

Term C: rate of change of B within CV as it is lost by fluid flow (Eulerian description)

$$\lim_{\Delta t \to 0} \frac{\text{Net amount of B leaving CV due to flow}}{\Delta t} = \text{rate of B leaving CV due to flow}$$
$$= \oint_{\mathrm{CS}} \rho\beta(\mathbf{u}\cdot\hat{\mathbf{n}})\ \mathrm{d}A,$$

where $(\mathbf{u}\cdot\hat{\mathbf{n}})$ quantifies the velocity component in the direction of the normal vector $\hat{\mathbf{n}}$.

By equating **Term A** = **Term B** + **Term C**,

$$\boxed{\frac{\mathrm{d}B_{\mathrm{system}}}{\mathrm{d}t} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\beta\ \mathrm{d}V + \oint_{\mathrm{CS}} \rho\beta(\mathbf{u}\cdot\hat{\mathbf{n}})\ \mathrm{d}A}, \tag{A.1}$$

which is the final expression of RTT.

In other words, $\begin{pmatrix}\text{Rate of change}\\ \text{of } B \text{ in the}\\ \text{system}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of } B \text{ in control}\\ \text{volume}\end{pmatrix} + \begin{pmatrix}\text{Net flux of } B \text{ out}\\ \text{of control volume}\end{pmatrix}.$

A.2 Conservation of Mass

For the following derivations, an infinitesimally small cube positioned in the Cartesian coordinate system is selected as the CV. The length of the edges is δ, hence, the coordinates of the two diagonal nodes are (x_0, y_0, z_0) and $(x_0+\delta, y_0+\delta, z_0+\delta)$, respectively. A surface on the cube has an area $A = \delta^2$, the cube has a volume $V = \delta^3$.

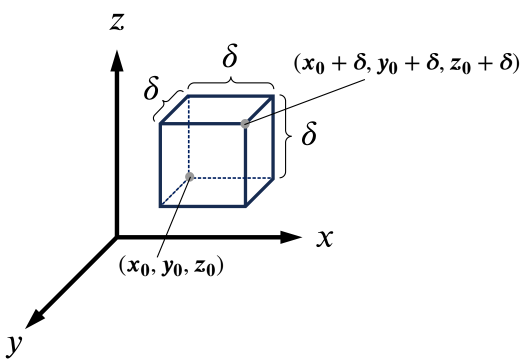


Figure A.2: Control volume used for the analysis.

Expressed in the language of RTT, mass conservation simply means the overall rate of change of mass is 0. Here, the physical quantity 'B' is mass, m; hence, following the definition, $\beta = \dfrac{\mathrm{d}B}{\mathrm{d}m} = \dfrac{\mathrm{d}m}{\mathrm{d}m} = 1$. Mathematically,

$$0 = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\ \mathrm{d}V + \oint_{\mathrm{CS}} \rho(\mathbf{u}\cdot\hat{\mathbf{n}})\ \mathrm{d}A. \tag{A.2}$$

To evaluate the first integral in Equation A.2, assume the variation of the density ρ is negligible within the CV, hence, $\int_{\mathrm{CV}} \rho \mathrm{d}V \approx \rho V = \rho\delta^3$. Differentiate the integral w.r.t. the time t,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\ \mathrm{d}V \approx \frac{\partial(\rho\delta^3)}{\partial t} = \delta^3\frac{\partial\rho}{\partial t}. \tag{A.3}$$

To evaluate the second integral in Equation A.2, we need to count the flow passing through the surfaces in three orthogonal directions. For the x-direction, the velocity component is u_x, the inlet and outlet surfaces are positioned at $x = x_0$ and $x = x_0 + \delta$, respectively; therefore,

$$\oint_{\mathrm{CS}} \rho u_x \, \mathrm{d}A = (\rho u_x \delta^2)\Big|_{x=x_0}^{x=x_0+\delta} = \delta^3 \left(\frac{(\rho u_x)|_{x=x_0+\delta} - (\rho u_x)|_{x=x_0}}{\delta} \right) \approx \delta^3 \frac{\partial(\rho u_x)}{\partial x}. \tag{A.4}$$

Similarly, for flow in the y-direction and z-direction, the surface integrals are

$$\oint_{\mathrm{CS}} \rho u_y \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho u_y)}{\partial y}, \tag{A.5}$$

$$\oint_{\mathrm{CS}} \rho u_z \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho u_z)}{\partial z}. \tag{A.6}$$

Combine Equation A.4, A.5, A.6,

$$\oint_{\mathrm{CS}} \rho(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A = \delta^3 \left(\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} \right), \tag{A.7}$$

or, in vector notation,

$$\oint_{\mathrm{CS}} \rho(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A = \delta^3 \nabla \cdot (\rho \mathbf{u}).$$

Substituting Equation A.3 and Equation A.7 into Equation A.2 yielding the celebrated continuity equation

$$0 \approx \delta^3 \frac{\partial \rho}{\partial t} + \delta^3 \nabla \cdot (\rho \mathbf{u}) \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}. \tag{A.8}$$

For incompressible fluid flow, the density ρ is constant, *i.e.*, it is invariant of time and space. This allows us to separate such a term from any partial derivatives in the equation, yielding the form

$$\nabla \cdot \mathbf{u} = 0. \tag{A.9}$$

A.3 Conservation of Linear Momentum

The linear momentum is the product between the mass and the velocity, $\mathbf{P} = m\mathbf{u}$. Hence, $\beta = \dfrac{\mathrm{d}\mathbf{P}}{\partial m} = \mathbf{u}$. By RTT, the conservation of linear momentum is

$$\mathbf{F} = \frac{\partial}{\partial t} \int_{\mathrm{CV}} \rho \mathbf{u} \, \mathrm{d}V + \oint_{\mathrm{CS}} \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A. \tag{A.10}$$

The L.H.S. of Equation A.10 is the total force exerted on the same CV as shown in Figure A.2. The total force can be further decomposed into

- The internal force that acts on the surfaces of the CV. It is comprised of the hydrostatic force that raises from the pressure load from the fluid flow; and the deviatoric force which is due to the fluid shear as the fluid moves with a velocity.

$$\mathbf{F}_{\text{internal}} = \underbrace{-V\nabla p}_{\text{hydrostatic}} + \underbrace{V\mu\nabla^2\mathbf{u}}_{\text{deviatoric}}.$$

  (Note that the “force” we mentioned here is force **per unit volume** $[\mathrm{N/m^3}]$ - hence, we multiply the volume to recover the actual force of the unit Newton.)

- The external force acting on CV that may be due to the presence of gravity, electromagnetism, *etc.*

$$\mathbf{F}_{\text{external}} = m\mathbf{f} = \rho V \mathbf{f}.$$

Hence, the total force

$$\mathbf{F} = \mathbf{F}_{\text{internal}} + \mathbf{F}_{\text{external}} = V(-\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}). \tag{A.11}$$

You may recognise the expression enclosed in the bracelet in Equation A.11 is the R.H.S. of the Navier-Stokes (N-S) momentum equation.

The first integral on the R.H.S. of Equation A.10 is evaluated following the same fashion as demonstrated in the derivation of mass conservation. Assume change of ρ and $\mathbf{u}$ is negligible within the CV, $\int\limits_{\text{CV}} \rho \mathbf{u} \mathrm{d}V \approx \rho \mathbf{u} V = \delta^3 \rho \mathbf{u}$, hence

$$\frac{\partial}{\partial t} \int\limits_{\text{CV}} \rho \mathbf{u} \, \mathrm{d}V \;\approx\; \frac{\partial(\delta^3 \rho \mathbf{u})}{\partial t} \;=\; \delta^3 \frac{\partial(\rho \mathbf{u})}{\partial t}. \tag{A.12}$$

The second integral on the R.H.S. of Equation A.10, we first consider the flow passing through the surfaces at the x-direction only, *i.e.*, $\mathbf{u} \cdot \hat{\mathbf{n}} = u_x$, leading to

$$\oint\limits_{\text{CS}} \rho \mathbf{u} u_x \, \mathrm{d}A = \rho \mathbf{u} u_x \delta^2 \Big|_{x=x_0}^{x=x_0+\delta} = \delta^3 \left(\frac{\rho \mathbf{u} u_x|_{x=x_0} - \rho \mathbf{u} u_x|_{x=x_0+\delta}}{\delta} \right) \approx \delta^3 \frac{\partial(\rho \mathbf{u} u_x)}{\partial x}. \tag{A.13}$$

Similarly, for flow in the y-direction and z-direction, the surface integrals are

$$\oint\limits_{\text{CS}} \rho \mathbf{u} u_y \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho \mathbf{u} u_y)}{\partial y}, \tag{A.14}$$

$$\oint\limits_{\text{CS}} \rho \mathbf{u} u_z \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho \mathbf{u} u_z)}{\partial z}. \tag{A.15}$$

Combine Equation A.13, A.14, A.15,

$$\oint\limits_{\text{CS}} \rho \mathbf{u} (\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A \approx \delta^3 \left(\frac{\partial(\rho \mathbf{u} u_x)}{\partial x} + \frac{\partial(\rho \mathbf{u} u_y)}{\partial y} + \frac{\partial(\rho \mathbf{u} u_z)}{\partial z} \right). \tag{A.16}$$

Substituting Equation A.11, Equation A.16, and Equation A.12 into Equation A.10, neglecting the common term $V = \delta^3$ from both sides, yielding the expression of the N-S momentum equation

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \frac{\partial(\rho \mathbf{u} u_x)}{\partial x} + \frac{\partial(\rho \mathbf{u} u_y)}{\partial y} + \frac{\partial(\rho \mathbf{u} u_z)}{\partial z} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}. \tag{A.17}$$

One final step to take is rearranging the unsteady and convective acceleration terms on the L.H.S. of Equation A.17. We can assume the fluid is incompressible, allowing us to separate ρ from the partial derivatives. Therefore,

$$\begin{aligned}
\frac{\partial(\rho \mathbf{u})}{\partial t} &= \rho \frac{\partial \mathbf{u}}{\partial t}, \\
\frac{\partial(\rho \mathbf{u} u_x)}{\partial x} &= \rho \frac{\partial(\mathbf{u} u_x)}{\partial x} = \rho \left(\frac{\partial u_x}{\partial x} \mathbf{u} + u_x \frac{\partial \mathbf{u}}{\partial x} \right), \\
\frac{\partial(\rho \mathbf{u} u_x)}{\partial x} &= \rho \frac{\partial(\mathbf{u} u_y)}{\partial y} = \rho \left(\frac{\partial u_y}{\partial y} \mathbf{u} + u_y \frac{\partial \mathbf{u}}{\partial y} \right), \\
\frac{\partial(\rho \mathbf{u} u_z)}{\partial z} &= \rho \frac{\partial(\mathbf{u} u_z)}{\partial z} = \rho \left(\frac{\partial u_z}{\partial z} \mathbf{u} + u_z \frac{\partial \mathbf{u}}{\partial z} \right).
\end{aligned}$$

Substitute the revised expressions into Equation A.17:

$$\cancel{\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)}\mathbf{u}+\rho\left(\frac{\partial \mathbf{u}}{\partial t}+u_x\frac{\partial \mathbf{u}}{\partial x}+u_y\frac{\partial \mathbf{u}}{\partial y}+u_z\frac{\partial \mathbf{u}}{\partial z}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+\rho\mathbf{f}, \tag{A.18}$$

or in compact form,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p+\mu\nabla^2\mathbf{u}+\rho\mathbf{f}. \tag{A.19}$$

which is the final expression of the N-S equation that we are all familiar with.