# Derivations of Navier-Stokes Continuity and Momentum Equations From Reynolds Transport Theorem

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## 1 Reynolds Transport Theorem

To elucidate the concept of the Reynolds Transport Theorem (RTT), we consider a control volume (CV) initially filled with a quantity $B$ that flows at a fixed speed $\mathbf{u}$. After some time, portions of $B$ initially inside the volume move outside and new portions of $B$ enter, as depicted by Figure 1.

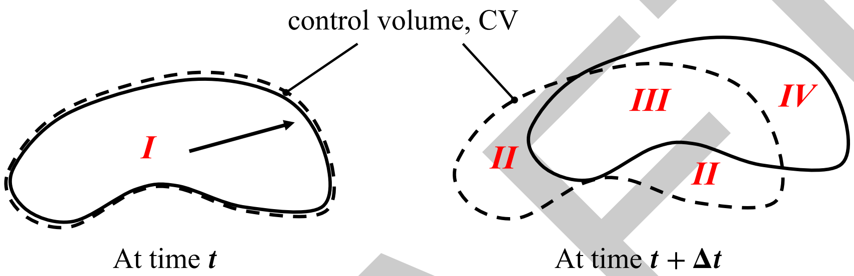


Figure 1: Movement of a physical quantity $B$ by fluid flow from inside to outside of the control volume.

Regions in Figure 1 are

- **I**: the entire fluid system within CV at time $t$
- **II**: new fluid that has entered CV at time $t + \Delta t$
- **III**: portion of fluid system that remains inside CV at time $t + \Delta t$
- **IV**: portion of a fluid system that is outside of CV at time $t + \Delta t$

By conservation of the quantity $B$ in the CV, *"how much out must be balanced by how much in"*,

$$\underbrace{B_{\text{system}}|_{t+\Delta t} - B_{\text{system}}|_{t}}_{\text{change of B in system}} = B_{III} + B_{IV} - B_{I}$$

$$= \underbrace{(B_{III} + B_{II} - B_{I})}_{\text{change of B in CV}} + \underbrace{(B_{IV} - B_{II})}_{\substack{\text{Net amount of B} \\ \text{leaving CV due to flow}}}$$

$$\underbrace{B_{\text{system}}|_{t+\Delta t} - B_{\text{system}}}_{\textbf{Term A}} = \underbrace{B_{CV}|_{t+\Delta t} - B_{CV}|_{t}}_{\textbf{Term B}} + \underbrace{\text{Net amount of B leaving CV due to flow}}_{\textbf{Term C}}$$

Divide each term by $\Delta t$, and limit the change in time to infinitesimally small: $\Delta t \to 0$.

**Term A**: rate of change of $B$ within the system (Lagrangian description)

$$\lim_{\Delta t \to 0} \frac{B_{\text{system}}|_{t+\Delta t} - B_{\text{system}}|_{t}}{\Delta t} = \frac{\mathrm{d}B_{\text{system}}}{\mathrm{d}t}.$$

**Term B**: rate of change of B within CV (Eulerian description)

$$\lim_{\Delta t \to 0} \frac{B_{CV}|_{t+\Delta t} - B_{CV}|_t}{\Delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t} \int_{CV} \rho \beta \, \mathrm{d}V,$$

where $\beta = \dfrac{\mathrm{d}B}{\mathrm{d}m}$ is the amount of $B$ per unit mass.

**Term C**: rate of change of B within CV as it is lost by fluid flow (Eulerian description)

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\text{Net amount of B leaving CV due to flow}}{\Delta t} &= \text{rate of B leaving CV due to flow} \\ &= \oint_{CS} \rho \beta (\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A, \end{aligned}$$

where $(\mathbf{u} \cdot \hat{\mathbf{n}})$ quantifies the velocity component in the direction of the normal vector $\hat{\mathbf{n}}$.

By equating **Term A** = **Term B** + **Term C**,

$$\boxed{\frac{\mathrm{d}B_{\text{system}}}{\mathrm{d}t} = \frac{\partial}{\partial t} \int_{CV} \rho \beta \, \mathrm{d}V + \oint_{CS} \rho \beta (\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A}, \tag{1.1}$$

which is the final expression of RTT.

In other words, $\begin{pmatrix} \text{Rate of change of} \\ B \text{ in the system} \end{pmatrix} = \begin{pmatrix} \text{Rate of change of} \\ B \text{ in control volume} \end{pmatrix} + \begin{pmatrix} \text{Net flux of } B \text{ out of} \\ \text{control volume} \end{pmatrix}.$

## 2 Conservation of Mass

For the following derivations, an infinitesimally small cube positioned in the Cartesian coordinate system is selected as the CV. The length of the edges is $\delta$, hence, the coordinates of the two diagonal nodes are $(x_0, y_0, z_0)$ and $(x_0 + \delta, y_0 + \delta, z_0 + \delta)$, respectively. A surface on the cube has an area $A = \delta^2$, the cube has a volume $V = \delta^3$.

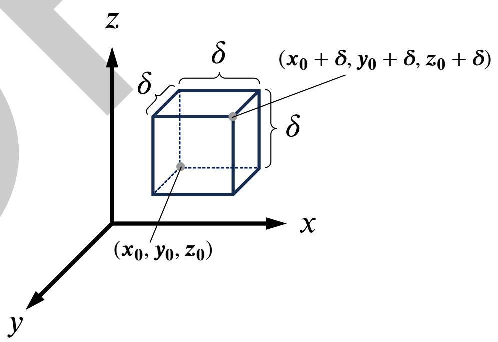


Figure 2: Control volume used for the analysis.

Expressed in the language of RTT, mass conservation simply means the overall rate of change of mass is 0. Here, the physical quantity '$B$' is mass, $m$; hence, following the definition, $\beta = \dfrac{\mathrm{d}B}{\mathrm{d}m} = \dfrac{\mathrm{d}m}{\mathrm{d}m} = 1$. Mathematically,

$$0 = \frac{\partial}{\partial t} \int_{CV} \rho \, \mathrm{d}V + \oint_{CS} \rho (\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A. \tag{2.1}$$

To evaluate the first integral in Equation 2.1, assume the variation of the density $\rho$ is negligible within the CV, hence, $\int_{CV} \rho \mathrm{d}V \approx \rho V = \rho\delta^3$. Differentiate the integral w.r.t. the time $t$,

$$\frac{\partial}{\partial t}\int_{CV} \rho \, \mathrm{d}V \approx \frac{\partial(\rho\delta^3)}{\partial t} = \delta^3 \frac{\partial \rho}{\partial t}. \tag{2.2}$$

To evaluate the second integral in Equation 2.1, we need to count the flow passing through the surfaces in three orthogonal directions. For the $x$-direction, the velocity component is $u_x$, the inlet and outlet surfaces are positioned at $x = x_0$ and $x = x_0 + \delta$, respectively; therefore,

$$\oint_{CS} \rho u_x \, \mathrm{d}A = (\rho u_x \delta^2)\Big|_{x=x_0}^{x=x_0+\delta} = \delta^3 \left( \frac{(\rho u_x)|_{x=x_0+\delta} - (\rho u_x)|_{x=x_0}}{\delta} \right) \approx \delta^3 \frac{\partial(\rho u_x)}{\partial x}. \tag{2.3}$$

Similarly, for flow in the $y$-direction and $z$-direction, the surface integrals are

$$\oint_{CS} \rho u_y \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho u_y)}{\partial y}, \tag{2.4}$$

$$\oint_{CS} \rho u_z \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho u_z)}{\partial z}. \tag{2.5}$$

Combine Equation 2.3, 2.4, 2.5,

$$\oint_{CS} \rho(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A = \delta^3 \left( \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} \right), \tag{2.6}$$

or, in compact notation,

$$\oint_{CS} \rho(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A = \delta^3 \nabla \cdot (\rho \mathbf{u}).$$

Substituting Equation 2.2 and Equation 2.6 into Equation 2.1 yielding the celebrated continuity equation

$$0 \approx \delta^3 \frac{\partial \rho}{\partial t} + \delta^3 \nabla \cdot (\rho \mathbf{u}) \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}. \tag{2.7}$$

For incompressible fluid flow, the density $\rho$ is constant, *i.e.*, it is invariant of time and space. This allows us to separate such a term from any partial derivatives in the equation, yielding the form

$$\nabla \cdot \mathbf{u} = 0. \tag{2.8}$$

# 3 Conservation of Linear Momentum

The linear momentum is the product between the mass and the velocity, $\mathbf{P} = m\mathbf{u}$. Hence, $\beta = \frac{\mathrm{d}\mathbf{P}}{\partial m} = \mathbf{u}$. By RTT, the conservation of linear momentum is

$$\mathbf{F} = \frac{\partial}{\partial t}\int_{CV} \rho \mathbf{u} \, \mathrm{d}V + \oint_{CS} \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A. \tag{3.1}$$

The L.H.S. of Equation 3.1 is the total force exerted on the same CV as shown in Figure 2. The total force can be further decomposed into

- The internal force that acts on the surfaces of the CV. It is comprised of the hydrostatic force that raises from the pressure load from the fluid flow; and the deviatoric force which is due to the fluid shear as the fluid moves with a velocity.

$$\mathbf{F}_{\text{internal}} = \underbrace{-V\nabla p}_{\text{hydrostatic}} + \underbrace{V\mu\nabla^2\mathbf{u}}_{\text{deviatoric}}.$$

  (Note that the "force" we mentioned here is force **per unit volume** $[\mathrm{N/m^3}]$ - hence, we multiply the volume to recover the actual force of the unit Newton.)

- The external force acting on CV that may be due to the presence of gravity, electromagnetism, *etc.*

$$\mathbf{F}_{\text{external}} = m\mathbf{f} = \rho V \mathbf{f}.$$

Hence, the total force

$$\mathbf{F} = \mathbf{F}_{\text{internal}} + \mathbf{F}_{\text{external}} = V(-\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}). \tag{3.2}$$

You may recognise the expression enclosed in the bracelet in Equation 3.2 is the R.H.S. of the Navier-Stokes (N-S) momentum equation.

The first integral on the R.H.S. of Equation 3.1 is evaluated following the same fashion as demonstrated in the derivation of mass conservation. Assume change of $\rho$ and $\mathbf{u}$ is negligible within the CV, $\int_{CV} \rho \mathbf{u} \mathrm{d}V \approx \rho \mathbf{u} V = \delta^3 \rho \mathbf{u}$, hence

$$\frac{\partial}{\partial t} \int_{CV} \rho \mathbf{u} \, \mathrm{d}V \approx \frac{\partial(\delta^3 \rho \mathbf{u})}{\partial t} = \delta^3 \frac{\partial(\rho \mathbf{u})}{\partial t}. \tag{3.3}$$

The second integral on the R.H.S. of Equation 3.1, we first consider the flow passing through the surfaces at the $x$-direction only, *i.e.*, $\mathbf{u} \cdot \hat{\mathbf{n}} = u_x$, leading to

$$\oint_{CS} \rho \mathbf{u} u_x \, \mathrm{d}A = \rho \mathbf{u} u_x \delta^2 \Big|_{x=x_0}^{x=x_0+\delta} = \delta^3 \left( \frac{\rho \mathbf{u} u_x|_{x=x_0} - \rho \mathbf{u} u_x|_{x=x_0+\delta}}{\delta} \right) \approx \delta^3 \frac{\partial(\rho \mathbf{u} u_x)}{\partial x}. \tag{3.4}$$

Similarly, for flow in the $y$-direction and $z$-direction, the surface integrals are

$$\oint_{CS} \rho \mathbf{u} u_y \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho \mathbf{u} u_y)}{\partial y}, \tag{3.5}$$

$$\oint_{CS} \rho \mathbf{u} u_z \, \mathrm{d}A \approx \delta^3 \frac{\partial(\rho \mathbf{u} u_z)}{\partial z}. \tag{3.6}$$

Combine Equation 3.4, 3.5, 3.6,

$$\oint_{CS} \rho \mathbf{u}(\mathbf{u} \cdot \hat{\mathbf{n}}) \, \mathrm{d}A \approx \delta^3 \left( \frac{\partial(\rho \mathbf{u} u_x)}{\partial x} + \frac{\partial(\rho \mathbf{u} u_y)}{\partial y} + \frac{\partial(\rho \mathbf{u} u_z)}{\partial z} \right). \tag{3.7}$$

Substituting Equation 3.2, Equation 3.7, and Equation 3.3 into Equation 3.1, neglecting the common term $V = \delta^3$ from both sides, yielding the expression of the N-S momentum equation

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \frac{\partial(\rho \mathbf{u} u_x)}{\partial x} + \frac{\partial(\rho \mathbf{u} u_y)}{\partial y} + \frac{\partial(\rho \mathbf{u} u_z)}{\partial z} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}. \tag{3.8}$$

One final step to take is rearranging the unsteady and convective acceleration terms on the L.H.S. of Equation 3.8. We can assume the fluid is incompressible, allowing us to separate $\rho$ from the partial derivatives. Therefore,

$$\begin{aligned}
\frac{\partial(\rho \mathbf{u})}{\partial t} &= \rho \frac{\partial \mathbf{u}}{\partial t}, \\
\frac{\partial(\rho \mathbf{u} u_x)}{\partial x} &= \rho \frac{\partial(\mathbf{u} u_x)}{\partial x} = \rho \left( \frac{\partial u_x}{\partial x} \mathbf{u} + u_x \frac{\partial \mathbf{u}}{\partial x} \right), \\
\frac{\partial(\rho \mathbf{u} u_x)}{\partial x} &= \rho \frac{\partial(\mathbf{u} u_y)}{\partial y} = \rho \left( \frac{\partial u_y}{\partial y} \mathbf{u} + u_y \frac{\partial \mathbf{u}}{\partial y} \right), \\
\frac{\partial(\rho \mathbf{u} u_z)}{\partial z} &= \rho \frac{\partial(\mathbf{u} u_z)}{\partial z} = \rho \left( \frac{\partial u_z}{\partial z} \mathbf{u} + u_z \frac{\partial \mathbf{u}}{\partial z} \right).
\end{aligned}$$

Substitute the revised expressions into Equation 3.8:

$$\cancel{\left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)} \mathbf{u} + \rho \left( \frac{\partial \mathbf{u}}{\partial t} + u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y} + u_z \frac{\partial \mathbf{u}}{\partial z} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}, \tag{3.9}$$

or in compact form,

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{f}. \tag{3.10}$$

which is the final expression of the N-S equation that we are all familiar with.