1.1 Tensors Analysis

Let

- ϕ denotes a scalar (0^{th}-order tensor), *e.g.*, density, viscosity.
- $\mathbf{f}$ (f_i or $\underline{f}$) denotes a vector (1^{st}-order tensor), *e.g.*, velocity.
- $\mathbf{T}$ (T_{ij} or $\underline{\underline{T}}$) denotes a matrix ($2^{\text{nd}}$-order tensor), *e.g.*, stress.

1. Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Properties:

$$\delta_{ij} x_j = x_i, \quad \delta_{ij} = \delta_{ji}$$

2. Alternating tensor (Levi-Civita):

$$\varepsilon_{ijk} = \begin{cases} 1 & \{i,j,k\} = \{1,2,3\}, \{2,3,1\}, \{3,1,2\} \\ -1 & \{i,j,k\} = \{3,2,1\}, \{2,1,3\}, \{1,3,2\} \\ 0 & \text{otherwise} \end{cases}$$

Properties:

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{kl}$$

$$\varepsilon_{ijk} = -\varepsilon_{ikj}$$

3. Dot product between two 1^{st}-order tensors

$$\mathbf{a} \cdot \mathbf{b} = a_i \, b_i$$

4. Cross product between two 1^{st}-order tensors

$$\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk} \, a_j \, b_k$$

5. Gradient of a 1^{st}-order tensor

$$(\nabla \mathbf{f})_{ij} = \frac{\partial f_i}{\partial x_j} = f_{i,j}$$

6. Gradient of a 2^{nd}-order tensor

$$(\nabla \mathbf{T})_{ijk} = \frac{\partial T_{jk}}{\partial x_i} = T_{jk,i}$$

7. Divergence of a 1^{st}-order tensor

$$(\nabla \cdot \mathbf{f})_i = \frac{\partial f_i}{\partial x_i} = f_{i,i}$$

8. Divergence of a 2^{nd}-order tensor

$$(\nabla \cdot \mathbf{T})_j = \frac{\partial T_{ij}}{\partial x_i} = T_{ij,i}$$

9. Curl of a 1^{st}-order tensor

$$(\nabla \times \mathbf{f})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} f_k = \varepsilon_{ijk} \, f_{k,j}$$

10. Curl of a 2^{nd}-order tensor

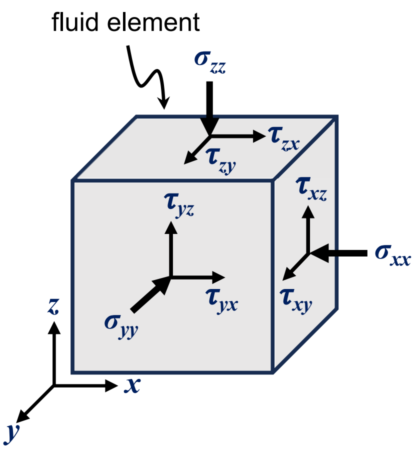
$$(\nabla \times \mathbf{T})_{ij} = \varepsilon_{ipq} \, T_{qj,p}$$

1.2 Constitutive Relationship for Fluids

1.2.1 Stress Tensor

1. In fluid mechanics, Cauchy stress tensor σ_{ij} describes the internal forces exerted on the fluid elements. It is comprised of the **hydrostatic** stress, $-p\delta_{ij}$, and the **deviatoric** stress, d_{ij},

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} = -p\delta_{ij} + d_{ij}.$$



2. Consider a fluid body at rest ($\mathbf{u} = 0$, absence of any shear forces), the only stress now acting on the fluid body is the **hydrostatic** stress, due to static pressure load from the fluid (Pascal's Law): $\sigma_{\text{hydrostatic}} = -p$.

The hydrostatic stresses correspond to the diagonal elements in the Cauchy stress tensor,

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\delta_{ij} \quad \Longrightarrow \quad p = \frac{1}{3}\,\mathrm{tr}(\sigma_{ij}).$$

3. The **deviatoric** (*a.k.a.* dynamic or viscous) stress raises when a fluid body is in motion. It can be approximated as a linear function of the rate of strain,
$$d_{ij} = \mathbb{C}_{ijkl}\ \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x}\right).$$
where $\mathbb{C}_{ijkl}$ is a 4$^{\text{th}}$-order tensor (for simplicity, think of it as a linear function coefficient, but it is not actually). Moreover, the rate of strain is equivalent to the velocity gradient,
$$\frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x}\right) = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial X}{\partial t}\right)}_{\nabla \mathbf{u}} \quad \Longrightarrow \quad d_{ij} = \mathbb{C}_{ijkl}\nabla\mathbf{u} = \mathbb{C}_{ijkl}\ \frac{1}{2}\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \cancelto{0,\ \text{neglect rotation}}{\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}\right].$$
Under *various* assumptions (material isotropy, tensor symmetry, and major symmetry), the number of combinations of $\mathbb{C}_{ijkl}$ can be reduced from 3^4=81 (4 free indices, each ranges 1-3) to 2. We have
$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}),$$
where λ and μ are the bulk viscosity (less significant, especially for incompressible fluid) and dynamics viscosity (more significant), respectively. To put all the facts together, the deviatoric stress
$$\begin{aligned} d_{ij} &= \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}) \times \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \\ &= \lambda\delta_{ij}\underbrace{\frac{\partial u_k}{\partial x_k}}_{\nabla\cdot\mathbf{u}} + \mu\underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{strain rate, } 2\mathbf{e}}. \end{aligned}$$

1.2.2 Strain Rate Tensor

Strain rate: $\mathbf{e} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)$	
in Cartesian coord. sys.	in cylindrical coord. sys.
$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & \frac{\partial v}{\partial y} & \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\ \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) & \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) & \frac{\partial w}{\partial z} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}) & \frac{1}{2}(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}) \\ \frac{1}{2}(r\frac{\partial(u_\theta/r)}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}) & \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} & \frac{1}{2}(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}) \\ \frac{1}{2}(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}) & \frac{1}{2}(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}) & \frac{\partial u_z}{\partial z} \end{pmatrix}$

1.2.3 Incompressible Fluid Constitutive Relationship

To put up all things together, the Cauchy stress tensor is
$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + d_{ij} \\ &= -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\ &= -p\mathbf{I} + \lambda(\nabla\cdot\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}. \end{aligned}$$

Cauchy's Equation For the incompressible fluid, $\frac{\partial u_k}{\partial x_k} = 0$ (mass conservation). Hence, $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

Cauchy's equation is obtained by equating the total forces acting on a fluid element to its acceleration, based on Newton's 2$^{\text{nd}}$ Law: $\mathbf{F} = m\mathbf{a}$.
$$\underbrace{\rho\frac{D\mathbf{u}}{Dt}}_{m\times\mathbf{a}} = \underbrace{\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{f}}_{\mathbf{F}_{\text{internal}} + \mathbf{F}_{\text{external}}},$$
where $\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}$ is the total (material) derivative. By expanding $\frac{D\mathbf{u}}{Dt}$ and $\nabla\cdot\boldsymbol{\sigma}$, we will obtain the celebrated Navier-Stokes equation, which depicts the conservation of linear momentum.