

2.1 Conservation Principles

Conservation of Momentum (Navier-Stokes)

(In compact/vector notation)
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

(In index notation)
$$\underbrace{\frac{\partial u_i}{\partial t}}_{\textcircled{1}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\textcircled{2}} = -\underbrace{\frac{1}{\rho} \frac{\partial p}{\partial x_i}}_{\textcircled{3}} + \underbrace{\nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{\textcircled{4}} + \underbrace{f_i}_{\textcircled{5}}$$

- ① rate of change of speed (unsteady)
- ② convective acceleration
- ③ pressure gradient
- ④ diffusion (viscous) acceleration
- ⑤ body force: gravitational, EM, etc.

Comments:

- Term ① and ② together represent the **total (material) derivative** of \mathbf{u} , which is the total acceleration of a fluid element. Term ① represents the acceleration in an *Eulerian* perspective, term ② represents the acceleration in a *Lagrangian* perspective.
- Term ③ and ④ represent the internal forces acting on a fluid element. Term ⑤ is the external force acting on a fluid element.
- The N-S is non-linear due to the presence of term ②; hence, N-S cannot be decomposed using basis functions (e.g. Fourier series).
- There are many possible formulations of the N-S equation. The abovementioned formulation assumes the fluid is incompressible (constant ρ) and Newtonian (constant μ , hence $\nu = \mu/\rho$ is also constant).

Other Transport Phenomena Transport of heat, mass, and momentum share similar mathematical frameworks.

Transport of ...	Governing Equation	“Diffusivity”	“Source”
Heat	$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \alpha \nabla^2 T + \dot{S}_T$	$\alpha = k/\rho c_p$	$\dot{S}_T = \dot{S}_v/\rho c_p$
Mass	$\frac{\partial C}{\partial t} + \mathbf{u} \cdot \nabla C = \mathcal{D} \nabla^2 C + S_C$	\mathcal{D}	S_C
Momentum (N-S)	$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u} + \dot{S}_v$	$\nu = \mu/\rho$	$\dot{S}_v = (-\nabla p + \rho \mathbf{f})/\rho$

2.2 The Navier-Stokes Equations

Cartesian Coordinates $\mathbf{u} \in [u, v, w]$

- Continuity Equation:

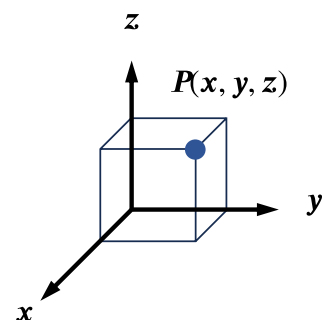
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Momentum Equations:

$$x : \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho f_x$$

$$y : \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho f_y$$

$$z : \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho f_z$$



Cylindrical Coordinates $\mathbf{u} \in [u_r, u_\theta, u_z]$

- Continuity Equation:

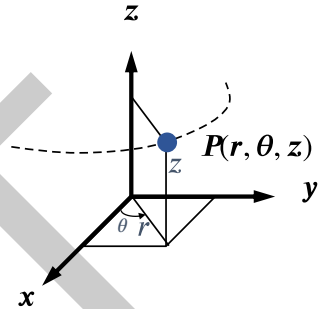
$$\frac{1}{r} \frac{\partial r u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

- Momentum Equations:

$$\begin{aligned} r : \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \right) \\ = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right] + \rho f_r \end{aligned}$$

$$\begin{aligned} \theta : \rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right] + \rho f_\theta \end{aligned}$$

$$\begin{aligned} z : \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) \\ = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] + \rho f_z \end{aligned}$$


Spherical Coordinates $\mathbf{u} \in [u_r, u_\theta, u_\phi]$

- Continuity Equation:

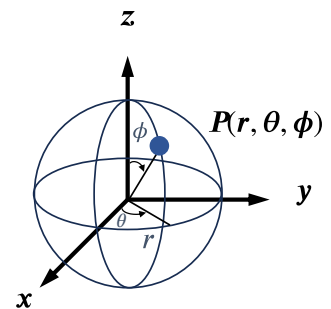
$$\frac{1}{r^2} \frac{\partial r^2 u_r}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial u_\theta \sin \theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} = 0$$

- Momentum Equations:

$$\begin{aligned} r : \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} \right) \\ = -\frac{\partial p}{\partial r} + \mu \left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{2}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right) + \rho f_r \end{aligned}$$

$$\begin{aligned} \theta : \rho \left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta - u_\phi^2 \cot \theta}{r} \right) \\ = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} \right) + \rho f_\theta \end{aligned}$$

$$\begin{aligned} \phi : \rho \left(\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi + u_\theta u_\phi \cot \theta}{r} \right) \\ = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \mu \left(\nabla^2 u_\phi - \frac{u_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \phi} \right) + \rho f_\phi \end{aligned}$$



2.2.1 Assumptions to Simplify the Navier-Stokes Equations

Assumption	Applicable to...	Mathematical expression	Note
Steady	Cartesian, cylindrical, spherical	$\frac{\partial}{\partial t} = 0$	
Fully developed	Cartesian, cylindrical, spherical	$\frac{\partial \mathbf{u}}{\partial n} = 0$	"Fully developed" indicates the velocity profile is independent of the location, not pressure.
Axisymmetric	cylindrical, spherical	$\frac{\partial}{\partial \theta} = 0$	
Spherical symmetric	spherical	$\frac{\partial}{\partial \theta} = 0, \frac{\partial}{\partial \phi} = 0$	
No swirl	cylindrical, spherical	$u_\theta = 0$	
Two-dimensional (with z-direction absent)	Cartesian	$u_z = 0, \frac{\partial}{\partial z} = 0$	For fluid flow in cylindrical coordinates, axisymmetric assumption simplifies the 3D flow to 2D flow.
Neglect body force	Cartesian, cylindrical, spherical	$\mathbf{f} = 0$	