2.1 Conservation Principles

Conservation of Momentum (Navier-Stokes)

(In compact/vector notation)
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

(In index notation)
$$\underbrace{\frac{\partial u_i}{\partial t}}_{①} + \underbrace{u_j\frac{\partial u_i}{\partial x_j}}_{②} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x_i}}_{③} + \underbrace{\nu\frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{④} + \underbrace{f_i}_{⑤}$$

① rate of change of speed (unsteady)
② convective acceleration
③ pressure gradient
④ diffusion (viscous) acceleration
⑤ body force: gravitational, EM, *etc.*

Comments:

- Term ① and ② together represent the **material derivative** of $\mathbf{u}$, which is the total acceleration of a fluid element. Term ① represents the acceleration in an *Eulerian* perspective, term ② represents the acceleration in a *Lagrangian* perspective.
- Term ③ and ④ represent the internal forces acting on a fluid element. Term ⑤ is the external force acting on a fluid element.
- The N-S is non-linear due to the presence of term ②; hence, N-S cannot be decomposed using basis functions (*e.g.* Fourier series).
- There are many possible formulations of the N-S equation. The abovementioned formulation assumes the fluid is incompressible (constant ρ) and Newtonian (constant μ, hence $\nu = \mu/\rho$ is also constant).

Other Transport Phenomena Transport of heat, mass, and momentum share similar mathematical frameworks.

Transport of ...	Governing Equation	"Diffusivity"	"Source"
Heat	$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \alpha\nabla^2 T + \dot{S}_T$	$\alpha = k/\rho c_p$	$\dot{S}_T = \dot{S}_v/\rho c_p$
Mass	$\frac{\partial C}{\partial t} + \mathbf{u}\cdot\nabla C = \mathscr{D}\nabla^2 C + S_C$	$\mathscr{D}$	S_C
Momentum (N-S)	$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\nabla^2\mathbf{u} + \dot{S}_v$	$\nu = \mu/\rho$	$\dot{S}_v = (-\nabla p + \rho\mathbf{f})/\rho$

2.2 The Navier-Stokes Equations

Cartesian Coordinates $\mathbf{u} \in [u, v, w]$

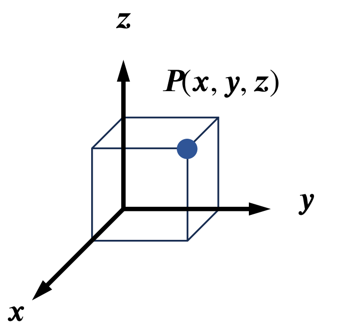
- Continuity Equation:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Momentum Equations:

$$\boldsymbol{x}: \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho f_x$$

$$\boldsymbol{y}: \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho f_y$$

$$\boldsymbol{z}: \quad \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho f_z$$



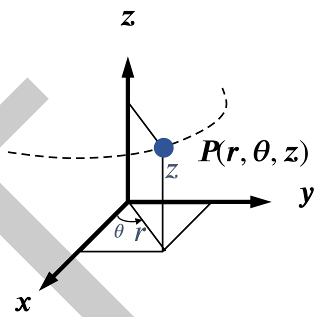
Cylindrical Coordinates $\mathbf{u} \in [u_r, u_\theta, u_z]$

- Continuity Equation:

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

- Momentum Equations:

$$\boldsymbol{r}: \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] + \rho f_r$$



$$\boldsymbol{\theta}: \rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] + \rho f_\theta$$

$$\boldsymbol{z}: \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho f_z$$

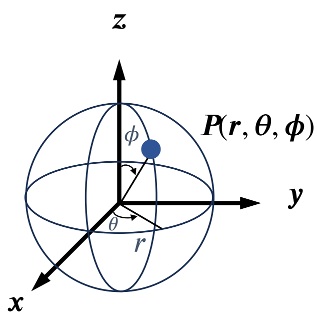
Spherical Coordinates $\mathbf{u} \in [u_r, u_\theta, u_\phi]$

- Continuity Equation:

$$\frac{1}{r^2}\frac{\partial r^2 u_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} = 0$$

- Momentum Equations:

$$\boldsymbol{r}: \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) + \frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right) + \rho f_r$$



$$\boldsymbol{\theta}: \rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{u_r u_\theta - u_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right) + \rho f_\theta$$

$$\boldsymbol{\phi}: \rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r u_\phi + u_\phi u_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial \phi}\right) + \rho f_\phi$$

2.2.1 Assumptions to Simplify the Navier-Stokes Equations

Assumption	**Applicable to...**	**Mathematical expression**	**Note**
Steady	Cartesian, cylindrical, spherical	$\frac{\partial}{\partial t} = 0$	
Fully developed	Cartesian, cylindrical, spherical	$\frac{\partial \mathbf{u}}{\partial n} = 0$	"Fully developed" indicates the velocity profile is independent of the location, not pressure.
Axisymmetric	cylindrical, spherical	$\frac{\partial}{\partial \theta} = 0$	
Spherical symmetric	spherical	$\frac{\partial}{\partial \theta} = 0, \quad \frac{\partial}{\partial \phi} = 0$	
No swirl	cylindrical, spherical	$u_\theta = 0$	
Two-dimensional (with z-direction absent)	Cartesian	$u_z = 0, \quad \frac{\partial}{\partial z} = 0$	For fluid flow in cylindrical coordinates, axisymmetric assumption simplifies the 3D flow to 2D flow.
Neglect body force	Cartesian, cylindrical, spherical	$\mathbf{f} = 0$	