5.1 Dimensional Analysis

Buckingham-Π Theorem The Buckingham-Π theorem states that if an equation involving k variables is dimensionally homogeneous (*i.e.*, L.H.S. units = R.H.S. units),

$$u_1 = f(u_2, u_3, ..., u_k),$$

it can be reduced to a relationship among $(k-r)$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables,

$$\Pi_1 = \phi(\Pi_2, \Pi_3, ...\Pi_{k-r}).$$

Example

Objective Perform the dimensional analysis of the scenario where the pressure drops per unit length along a smooth pipe.

Step 1 List all relevant variables in the objective equation to be non-dimensionalised. Here,

$$\Delta p_l = f(D, \rho, \mu, V),$$

where the pressure drop Δp_l is a function of the pipe diameter D, the density ρ, the (dynamic) viscosity μ, and velocity V.

Step 2 List the dimensions of the variables. Let $[M]$ denotes the dimension of mass, $[L]$ denotes the dimension of length, $[T]$ denotes the dimension of time, (refer to Table 2)

$$\begin{aligned} \Delta p_l &\doteq [ML^{-1}T^{-2}], & \mu &\doteq [ML^{-1}T^{-1}] \\ D &\doteq [L], & V &\doteq [LT^{-1}] \\ \rho &\doteq [ML^{-3}] \end{aligned}$$

There are $k = 5$ variables and $r = 3$ reference dimensions, we conclude there will be $k - r = 2$ dimensionless groups.

Step 3 Suppose the first group involves Δp_l, ρ, V and D. Let a, b, c, d denote 4 constants to be determined,

$$D^a \rho^b V^c \Delta p_l^d \implies \boxed{[L]^a [ML^{-3}]^b [LT^{-1}]^c [ML^{-1}T^{-2}]^d \doteq [L]^0 [F]^0 [T]^0}.$$

Balance of $[M]$, $[L]$, $[T]$ would give the simultaneous equations

$$\begin{aligned} \text{(mass)} \quad & b + d = 0, \\ \text{(length)} \quad & a - 3b + c - d = 0, \\ \text{(time)} \quad & -c - 2d = 0. \end{aligned}$$

(3 equations with 4 unknowns ⇒ the equation system is underdetermined, we will not be able to explicitly solve the numerical values of 4 parameters, but at least we will know the relations between a, b, c, d.)

resulting in the following relations: $a = 0$, $b = -d$, $c = -2d$. Hence, with $d = -1$, $\longrightarrow$ $a = 0$, $b = 1$, $c = 2$,

$$D^0 \rho^1 V^2 \Delta p_l^{-1} \equiv \left(\frac{\rho V^2}{\Delta p_l}\right) \text{ is dimensionless,} \implies \boxed{\Pi_1 = \left(\frac{\rho V^2}{\Delta p_l}\right)}.$$

(Although we *supposed* that D might get involved in the first Π group, but by $a = 0$, Π_1 is invariant of D.)

Step 4 Similarly, the second term involves μ, follow the same rule, this yields $\boxed{\Pi_2 = \frac{\mu}{\rho D V}}$, which is $1/\mathrm{Re}$.

Step 5 Hence, we can express the result of the dimensional analysis as

$$\frac{\rho V^2}{\Delta p_l} = \phi\left(\frac{\mu}{\rho D V}\right).$$

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, L; Density, ρ; Frequency of oscillating flow, ω; Pressure, p; Speed of sound, c; Surface tension, σ_s ; Velocity, V.

Dimensionless groups	Name	Interpretation	Types of Applications
$\rho V L/\mu$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$V/\sqrt{gL}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$p/\rho V$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
V/c	Mach number, Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\omega L/V$	Strouhal number, St	$\frac{\text{inertia(local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\rho V^2 L/\sigma_s$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

Table 1: Common variables and dimensionless groups in fluid mechanics.

Parameter	Symbol	Dimensions	Parameter	Symbol	Dimensions
Acceleration	a	$[L^1T^{-2}]$	Surface tension	σ_s	$[M^1T^{-2}]$
Angle	θ, ϕ, etc.	1 (none)	Velocity	V	$[L^1T^{-1}]$
Density	ρ	$[M^1L^{-3}]$	Viscosity	μ	$[M^1L^{-1}T^{-1}]$
Force	F	$[M^1L^1T^{-2}]$	Volume flow rate	Q	$[L^3T^{-1}]$
Frequency	f	$[T^{-1}]$	Pressure	p	$[M^1L^{-1}T^{-2}]$

Table 2: Table of parameters with symbols and primary dimensions in two columns. $[M]$: mass, $[T]$: time; $[L]$: length.

5.2 Non-Dimensional Navier-Stokes Equation

- Define the non-dimensional variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \qquad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \qquad t^* = \frac{t}{L/U}, \qquad p^* = \frac{p}{P_0},$$

  where L, U are the characteristic length and velocity, respectively.

- The dimensionless Navier-Stokes momentum equation is

$$\mathrm{Re}\Big(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\Big) = -\frac{P_0}{\frac{\mu U}{L}}\nabla^* p^* + \nabla^{*2}\mathbf{u}^*,$$

  where $P_0 = \dfrac{\mu U}{L}\max(1, \mathrm{Re})$, *i.e.*, the viscous scale ($\mathrm{Re} < 1$) or dynamic scale ($\mathrm{Re} > 1$). This formulation ensures the pressure term has the same order of magnitude as other terms, since there is no natural scaling for pressure.

- The dimensionless continuity equation is

$$\nabla^* \cdot \mathbf{u}^* = 0.$$

Small Re **flow (**$\mathrm{Re} \ll 1$**)** $P_0 = \mu U/L$ and the L.H.S. eliminated,

$$\cancelto{0}{\mathrm{Re}\Big(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\Big)} = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^* \quad \Longrightarrow \quad \nabla^* p^* = \nabla^{*2}\mathbf{u}^* \quad \Longleftrightarrow \quad \mu\nabla^2\mathbf{u} = \nabla p$$

which is known as the **Stokes equation** that can be solved analytically due to its linearity.

Governing Equation of Stokes Flow

Define the vorticity as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$

$$\mu \nabla^2 \mathbf{u} = -\mu \nabla \times \boldsymbol{\omega} \qquad \text{due to} \qquad \nabla \times \boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{u}) = \cancel{\nabla \cdot \mathbf{u}} - \nabla^2 \mathbf{u}.$$

Further, take the curl of $\mu \nabla^2 \mathbf{u} = \nabla p$:

$$\underbrace{\nabla \times \nabla p}_{\text{“curl of grad is zero”}} = \nabla \times (\mu \nabla^2 \mathbf{u}) \implies \begin{aligned} 0 &= -\mu \nabla \times (\nabla \times \boldsymbol{\omega}) \\ 0 &= -\mu [\underbrace{\nabla(\nabla \cdot \boldsymbol{\omega}) - \nabla^2 \boldsymbol{\omega}}_{\text{by: } \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}}] \\ 0 &= -\mu [\underbrace{\nabla(\nabla \cdot \nabla \times \mathbf{u})}_{\text{“div of curl is zero”}} - \nabla^2 \boldsymbol{\omega}]. \end{aligned}$$

The above derivation results in $\nabla^2 \boldsymbol{\omega} = 0$, which is the governing equation of the Stokes flow.

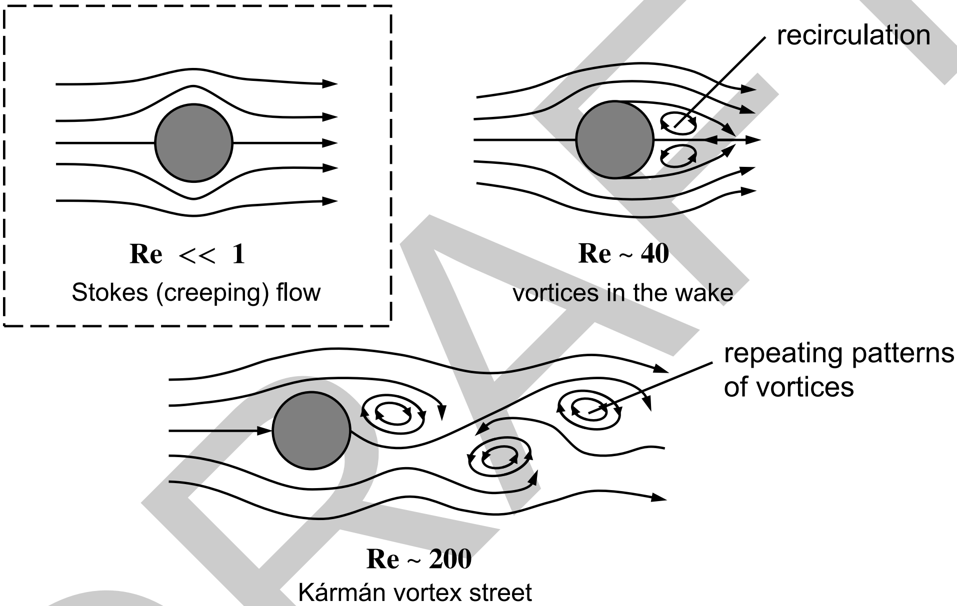

FIG. 1: Flow passing around a cylinder at different Reynolds numbers. The top left scenario depicts the Stokes flow when $\mathrm{Re} \ll 1$ - no flow separation.

Large Re **flow (**$\mathrm{Re} \gg 1$**)** $P_0 = \rho U^2$ and the viscus term eliminated (hence, the fluid is approximated nearly inviscid),

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* = -\nabla^* p^* \implies \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p,$$

which is known as the **Euler equation.** (see Supplementary 5 for boundary layer analysis.)

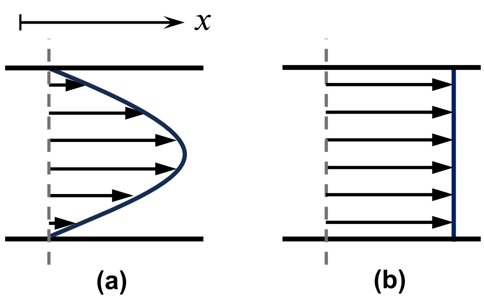

FIG. 2: The velocity profile of flow between two parallel plates when the fluid is (a) affected by viscosity, (b) inviscid.