6.1 Finite Difference Method: An Example

Consider the following example boundary value problem:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} = 0 \quad \text{with} \quad \begin{cases} u = 1, & @x = 0 \\ u = 0, & @x = 1 \end{cases}.$$

Analytical Solution Procedure

Let $u = e^{rx}$, hence $u' = re^{rx}$ and $u'' = r^2 e^{rx}$, substitute this relation back to the governing equation, we get

$$\begin{aligned} 0 &= r^2 e^{rx} + 2re^{rx} \\ &= (r^2 + 2r)e^{rx} \end{aligned}$$

Hence, $r_1 = 0 \Rightarrow u = 1$ and $r_2 = -2 \Rightarrow u = e^{-2x}$,

$$u = A + Be^{-2x},$$

where A and B are two constants to be determined from the boundary conditions. Apply the boundary conditions,

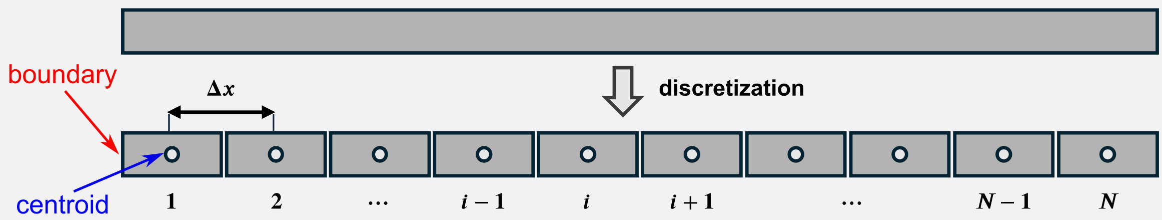
$$\begin{cases} A + B = 1 \\ A + e^{-2}B = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} A = -\frac{e^{-2}}{1-e^{-2}} \\ B = \frac{1}{1-e^{-2}} \end{cases}.$$

Hence, the general solution is

$$u(x) = -\frac{e^{-2}}{1 - e^{-2}} + \frac{1}{1 - e^{-2}} e^{-2x}.$$

Numerical Solution Procedure

First, we **discritize** the entire continuous domain into a finite number of N grids, with a constant distance between two adjacent grids being Δx (practically, of your own choice).



Let $u(x_i + \Delta x)$ and $u(x_i - \Delta x)$ denote the value of u at *next* grid and the *previous grid* in relation to the *current* grid x_i. We can use Taylor series expansion to find the value of $u(x_i + \Delta x)$ and $u(x_i - \Delta x)$ in terms of $u(x_i)$,

$$u(x_i - \Delta x) = u(x_i) - u'(x_i)\Delta x + \frac{1}{2}u''(x_i)\Delta x^2 - \frac{1}{6}u'''(x_i)\Delta x^3 + \mathcal{O}(\Delta x^4) \tag{1}$$

$$u(x_i + \Delta x) = u(x_i) + u'(x_i)\Delta x + \frac{1}{2}u''(x_i)\Delta x^2 + \frac{1}{6}u'''(x_i)\Delta x^3 + \mathcal{O}(\Delta x^4) \tag{2}$$

Equation (1) + (2) $\Rightarrow$ we will get the expression of the second order derivative term, (neglect the H.O.T.)

$$u(x_i - \Delta x) + u(x_i + \Delta x) = 2u(x_i) + u''(x_i)\Delta x^2 + \mathcal{O}(\Delta x^4)$$

$$\Rightarrow u''(x_i) = \frac{1}{\Delta x^2}[u(x_i - \Delta x) - 2u(x_i) + u(x_i + \Delta x)] + \cancel{\mathcal{O}(\Delta x^2)}$$

Equation (2) - (1) $\Rightarrow$ we will get the expression of the first order derivative term, (neglect the H.O.T.)

$$u(x + \Delta x) - u(x - \Delta x) = 2u'(x)\Delta x + \frac{1}{3}u'''(x)\Delta x^3 + \mathcal{O}(\Delta x^4)$$

$$\Rightarrow u'(x_i) = \frac{1}{2\Delta x}[u(x_i + \Delta x) - u(x_i - \Delta x)] + \cancel{\mathcal{O}(\Delta x^2)}$$

The method shown above is commonly referred to as the **central differencing scheme**, which has a 2nd-order accuracy.

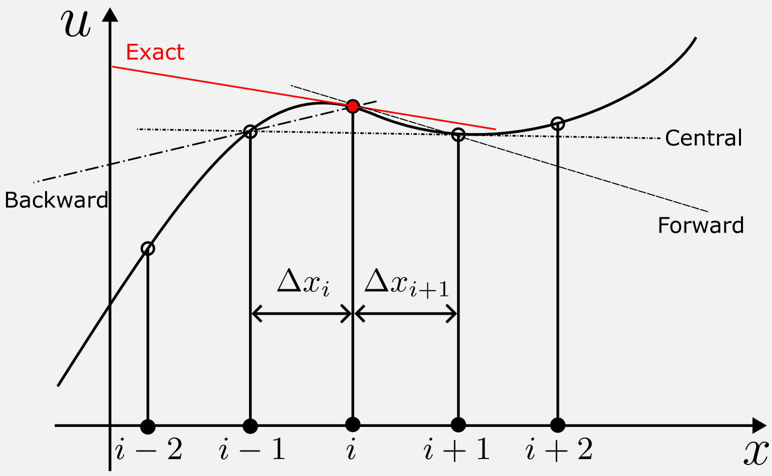


FIG. 1: A graphical illustration of approximating the first-order derivative $\mathrm{d}u_i/\mathrm{d}x$ using forward, backward, and central differencing schemes.

Hence, the governing equation

$$
\begin{aligned}
&\Rightarrow \quad \frac{1}{\Delta x^2}[u(x_i-\Delta x)-2u(x_i)+u(x_i+\Delta x)] + \frac{2}{2\Delta x}[u(x_i+\Delta x)-u(x_i-\Delta x)] = 0 \\
&\Rightarrow \quad \left(\frac{1}{\Delta x^2}-\frac{1}{\Delta x}\right)u(x_i-\Delta x) + \left(-\frac{2}{\Delta x^2}\right)u(x_i) + \left(\frac{1}{\Delta x^2}+\frac{1}{\Delta x}\right)u(x_i+\Delta x) = 0
\end{aligned}
$$

For the index i ranges from 1 to $N-1$, the above expression can be converted into the matrix form $\mathbf{A}\mathbf{u} = \mathbf{b}$,

$$
\underbrace{\begin{pmatrix}
(-\frac{2}{\Delta x^2}) & (\frac{1}{\Delta x^2}+\frac{1}{\Delta x}) & 0 & 0 & \dots & 0 & 0 \\
(\frac{1}{\Delta x^2}-\frac{1}{\Delta x}) & (-\frac{2}{\Delta x^2}) & (\frac{1}{\Delta x^2}+\frac{1}{\Delta x}) & 0 & \dots & 0 & 0 \\
0 & (\frac{1}{\Delta x^2}-\frac{1}{\Delta x}) & (-\frac{2}{\Delta x^2}) & (\frac{1}{\Delta x^2}+\frac{1}{\Delta x}) & \dots & 0 & 0 \\
0 & 0 & (\frac{1}{\Delta x^2}-\frac{1}{\Delta x}) & (-\frac{2}{\Delta x^2}) & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & \dots & (-\frac{2}{\Delta x^2}) & (\frac{1}{\Delta x^2}+\frac{1}{\Delta x}) \\
0 & 0 & 0 & 0 & \dots & (\frac{1}{\Delta x^2}-\frac{1}{\Delta x}) & (-\frac{2}{\Delta x^2})
\end{pmatrix}}_{\mathbf{A}}
\underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix}}_{\mathbf{u}}
=
\underbrace{\begin{pmatrix} -\frac{1}{\Delta x^2}+\frac{1}{\Delta x} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{b}},
$$

and $\mathbf{u}$ is solvable by finding $\mathbf{A}^{-1}\mathbf{b}$.

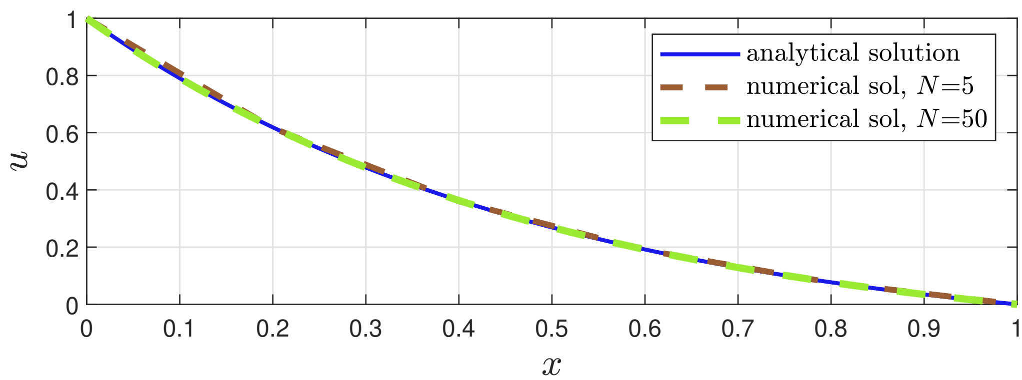


FIG. 2: Comparison of the analytical solution and the numerical solutions (discretized with $N = 5$ and $N = 50$) of the same governing equation.