Any notes/figures demonstrated below are self-contained and included here for the purpose of completeness - some of which may be absent from the main lecture materials. Please review with discretion.

S1.1 Proofs and Verifications of Some Properties Seen in Tensor Analysis

1. Proof of $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{kp}\delta_{jq}$

From the lecture, you have seen the contraction property of two Levi-Civita symbols sharing one same index is $\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{kp}\delta_{jq}$. The following section shall explore the derivation of this property.

We first introduce three orthonormal bases

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

With the three orthogonal bases defined above, we can re-write the Kronecker delta as

$$\delta_{ij} = \mathbf{e}_i^{\top}\mathbf{e}_j,$$

where i and j are two free indices ranging from 1 to 3.

Examples to verify $\delta_{ij} = \mathbf{e}_i^{\top}\mathbf{e}_j$

- For the case $i = 1$, $j = 1$,

$$\mathbf{e}_i^{\top}\mathbf{e}_j = \mathbf{e}_1^{\top}\mathbf{e}_2 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = 1 \quad \equiv \quad \delta_{11} = 1 \quad ✓$$

- For the case $i = 1$, $j = 2$,

$$\mathbf{e}_i^{\top}\mathbf{e}_j = \mathbf{e}_1^{\top}\mathbf{e}_2 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \quad \equiv \quad \delta_{12} = 0 \quad ✓$$

Same for other combinations of i and j.

Similarly, we can re-write the Levi-Civita symbol as

$$\varepsilon_{ijk} = |\mathbf{e}_i \ \mathbf{e}_j \ \mathbf{e}_k|,$$

where $|\star|$ denotes the determinant of the matrix.

Examples to verify $\varepsilon_{ijk} = |\mathbf{e}_i \ \mathbf{e}_j \ \mathbf{e}_k|$

- For the case $i = 1$, $j = 1$, $k = 3$,

$$|\mathbf{e}_i \ \mathbf{e}_j \ \mathbf{e}_k| = |\mathbf{e}_1 \ \mathbf{e}_1 \ \mathbf{e}_3| = \begin{vmatrix}1 & 1 & 0\\0 & 0 & 0\\0 & 0 & 1\end{vmatrix} = 0 \quad \equiv \quad \varepsilon_{113} = 0 \quad ✓$$

- For the case $i = 1$, $j = 2$, $k = 3$,

$$|\mathbf{e}_i \ \mathbf{e}_j \ \mathbf{e}_k| = |\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3| = \begin{vmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{vmatrix} = 1 \quad \equiv \quad \varepsilon_{123} = 1 \quad ✓$$

Same for other combinations of i, j, k.

With the identities above, the product of two Levi-Civitas (sharing the same 'i' index) are

$$\varepsilon_{ijk}\varepsilon_{ipq} = \begin{vmatrix}\mathbf{e}_i & \mathbf{e}_j & \mathbf{e}_k\end{vmatrix}\begin{vmatrix}\mathbf{e}_i^{\top}\\ \mathbf{e}_p^{\top}\\ \mathbf{e}_q^{\top}\end{vmatrix} = \begin{vmatrix}\mathbf{e}_i^{\top}\mathbf{e}_i & \mathbf{e}_i^{\top}\mathbf{e}_j & \mathbf{e}_i^{\top}\mathbf{e}_k\\ \mathbf{e}_p^{\top}\mathbf{e}_i & \mathbf{e}_p^{\top}\mathbf{e}_j & \mathbf{e}_p^{\top}\mathbf{e}_k\\ \mathbf{e}_q^{\top}\mathbf{e}_i & \mathbf{e}_q^{\top}\mathbf{e}_j & \mathbf{e}_q^{\top}\mathbf{e}_k\end{vmatrix} = \begin{vmatrix}\delta_{ii} & \delta_{ij} & \delta_{ik}\\ \delta_{pi} & \delta_{pj} & \delta_{pk}\\ \delta_{qi} & \delta_{qj} & \delta_{qk}\end{vmatrix}.$$

For the case $i \neq j \neq k$, we know $\delta_{ii} = 1$, $\delta_{ij} = 0$, $\delta_{ik} = 0$, leading the determinant simplified as

$$\varepsilon_{ijk}\varepsilon_{ipq} = \begin{vmatrix} 1 & 0 & 0 \\ \delta_{pi} & \delta_{pj} & \delta_{pk} \\ \delta_{qi} & \delta_{qj} & \delta_{qk} \end{vmatrix} = 1 \cdot (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{jq}),$$

which proves the contraction property as desired.

2. **Verify** $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$ by exhaustion

Let the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. By the right hand rule, the product of $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.
From the lecture, you have seen $\mathbf{a} \times \mathbf{b}$ can also be expressed in tensor notation: $(\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$. The following section shall explore why this formula works by exhausting the possible values of indices.

Each index i, j, k can freely range from 1 to 3. For the case $i = 1$, by $(\mathbf{a} \times \mathbf{b})_{i=1} = \varepsilon_{1jk} a_j b_k$, we can list all combinations of $j = \{1, 2, 3\}$ and $k = \{1, 2, 3\}$

	$j = 1$	$j = 2$	$j = 3$
$k = 1$	$\varepsilon_{111} a_1 b_1$	$\varepsilon_{121} a_2 b_1$	$\varepsilon_{131} a_3 b_1$
$k = 2$	$\varepsilon_{112} a_1 b_2$	$\varepsilon_{122} a_2 b_2$	$\varepsilon_{132} a_3 b_2$
$k = 3$	$\varepsilon_{113} a_1 b_3$	$\varepsilon_{123} a_2 b_3$	$\varepsilon_{133} a_3 b_3$

Recall the definition of the Levi-Civita symbol: $\varepsilon_{ijk} = \begin{cases} 1, & \{i,j,k\} = \{1,2,3\}, \{2,3,1\}, \{3,1,2\} \\ -1, & \{i,j,k\} = \{3,2,1\}, \{2,1,3\}, \{1,3,2\} \\ 0, & \text{otherwise} \end{cases}$. Hence, the only two non-zero terms left in the table above are $\varepsilon_{132} = -1$ and $\varepsilon_{123} = 1$,

	$j = 1$	$j = 2$	$j = 3$
$k = 1$	0	0	0
$k = 2$	0	0	$-a_3 b_2$
$k = 3$	0	$a_2 b_3$	0

Summing these two terms: $(\mathbf{a} \times \mathbf{b})_{i=1} = a_2 b_3 - a_3 b_2$, **which is the 1$^{\text{st}}$ component of the $\mathbf{a} \times \mathbf{b}$ vector**.

Following the same procedure to exhaust the combinations of j and k under $i = 2$ and $i = 3$ surely yields the result that 'coincides' with the i^{th} vector components in $\mathbf{a} \times \mathbf{b}$.

3. **Proof of** $\nabla \times \nabla \phi = 0$

By writing $\nabla \times \nabla \phi$ in index notation,

$$\varepsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \phi)_k = \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} \tag{A}$$

Swipe the notation in equation A: $j \leftrightarrow k$

$$\varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} = \varepsilon_{ikj} \frac{\partial^2 \phi}{\partial x_k \partial x_j} \tag{B}$$

Also, by the commutative property of partial differentiation, equation A can be written as

$$\varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k} = \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_k \partial x_j} \tag{C}$$

Equate equation B and C: $\varepsilon_{ikj} \frac{\partial^2 \phi}{\partial x_k \partial x_j} = \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial x_k \partial x_j}$. Since $\varepsilon_{ijk} = -\varepsilon_{ikj}$, the only circumstance to equate the L.H.S. and R.H.S. is both terms are 0. (*cf.* $-a = a$ indicates $a = 0$). Hence, $\nabla \times \nabla \phi = 0$.