Any notes/figures demonstrated below are self-contained and included here for the purpose of completeness - some of which may be absent from the main lecture materials. Please review with discretion.

S4.1 Turbulent Boundary Layer

What is Boundary Layer? The boundary layer (BL) is a thin layer of fluid in the vicinity of the wall in which the velocity raises from zero at the wall surface (non-slip) to the freestream velocity (*i.e.*, U in Figure 1) away from the surface (along the y-direction in Figure 1). Outside the boundary layer, the mean flow velocity equals U.

BLs can be laminar, transition, or turbulent. As depicted in Figure 1, the laminar BL is a very smooth flow; the turbulent BL contains swirls (eddies); while the transition boundary layer is in between.

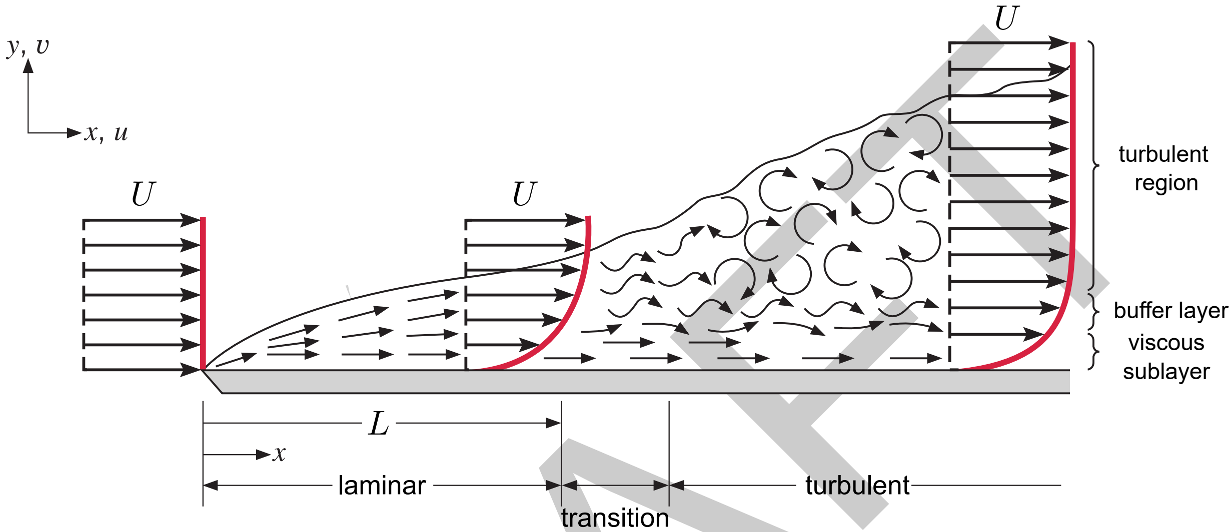

FIG. 1: Velocity boundary layer development on a flat plate. (Incropera *et al.*)

Turbulent Boundary Layer Computational modelling of turbulent flow is particularly challenging in the near-wall region - predictions of the near-wall flow characteristics are inaccurate for algorithms developed for the free shear flow. Hence, by understanding the turbulent BL approximations, wall functions can be coupled to these algorithms facilitating a special near-wall treatment.

Define $y^+ = y\dfrac{u^*}{\nu}$, $u^+ = \dfrac{\bar{u}}{u^*}$, where $u^* = \sqrt{\tau_w/\rho}$ is termed the friction velocity, ν is the kinematic viscosity, $\bar{u}$ is the mean turbulence velocity. The plot of u^+ versus y^+ is shown in Figure 2.

Layer	Range of y^+	Relation of y^+ & u^+
viscous sublayer	$0 < y^+ < 5 \sim 8$	$u^+ \approx y^+$
buffer layer	$5 \sim 8 < y^+ < 30 \sim 70$	(blended)
overlap (inner) layer	$30 \sim 70 < y^+ < 10^4$	$u^+ = 1/\kappa \ln y^+ + B$
outer (wake) region	$y^+ > 10^4$	not strictly defined

(κ and B are both emperical constants.)

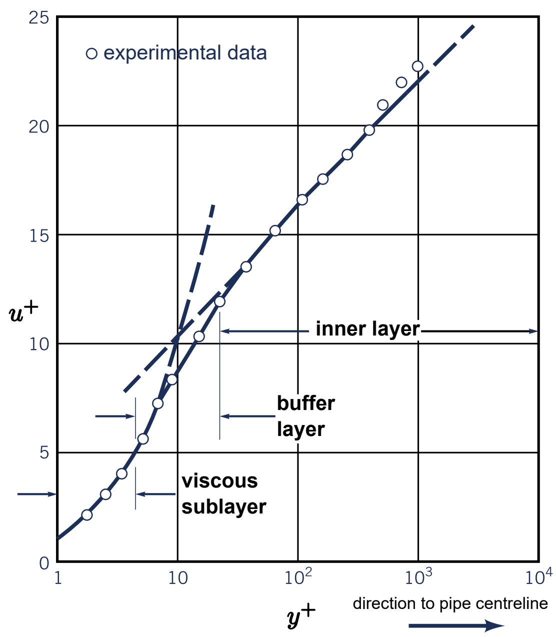

FIG. 2: Typical structure of the turbulent velocity profile in a pipe. (Munson *et al.*)

S4.2 Energy Cascade in Turbulent Flow

The progression, or breakdown of large eddies to small eddies is referred to as the **energy cascade** (see Summary 4, Figure 1). Let k $(= \frac{1}{2}\overline{u_i' u_i'})$ denotes the turbulence kinetic energy, ε $(= \mathrm{d}k/\mathrm{d}t)$ denotes the dissipation rate of turbulence

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kinetic energy,

- **Integral scale**: the largest scale where the turbulence kinetic energy is generated. It can be related to the size of the system (*e.g.*, 10% ∼ 20% of the pipe diameter). The integral length L and time τ_L scales are

$$L = \frac{k^{\frac{3}{2}}}{\varepsilon}, \qquad \tau_L = \frac{k}{\varepsilon}.$$

- **Kolmogorov scale**: the smallest scale which measures the size of the smallest eddies in the flow regime. This is where the turbulent kinetic energy dissipated *via* the molecular viscosity. The Kolmogorov length η and time τ_η scales are

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}, \qquad \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}}.$$

- **Taylor scale**: the intermediate scale between the integral scale and Kolmogorov scale.

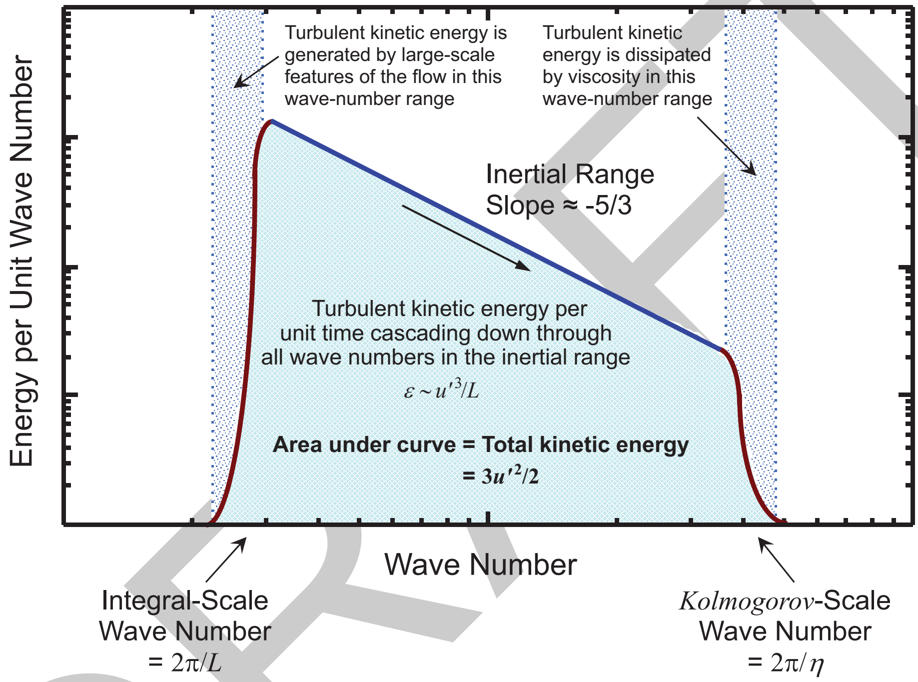

FIG. 3: Turbulence kinetic energy cascade as a function of wave number (∼ 1/size of eddies). (P. Aleiferis.)

S4.3 Closure

One key problem that remains unanswered is the velocity fluctuation $\overline{u_i' u_j'}$ that appears in the time-averaged form of the momentum equation is an unknown term. How do we model the turbulence then? Obviously, we need an expression that bridges such unknown quantities to the known quantities, and this is referred to as the closure problem.

Joseph Valentin Boussinesq (1842-1929) proposed that instead of finding the Reynolds stresses, one can alternatively find the turbulent viscosity. According to his theory, the Reynolds stress and the turbulent viscosity are linked through

$$\overline{u_i' u_j'} = \frac{2}{3}\delta_{ij} k - \nu_t \left[\frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_j}\right],$$

where ν_t is the turbulent viscosity; note that ν_t is a property of the *turbulent flow*, not the fluid (whereas the kinematic viscosity ν is a property of fluid). This is known as the **Boussinesq approximation**.

The standard k-ϵ turbulence model (Launder *et. al.*, 1969), is a good example of turbulence modelling based on the Boussinesq approximation. In the solution procedure, the turbulent viscosity is calculated from the empirical relations, following two additional conservation equations - one for turbulent KE k, one for turbulence kinetic energy dissipation rate ϵ, are solved in addition to the continuity and momentum equation.

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