Any notes/figures demonstrated below are self-contained and included here for the purpose of completeness - some of which may be absent from the main lecture materials. Please review with discretion.

S5.1 Boundary Layer Approximation

Motivation Albeit the N-S equation has been formulated early since the mid-1800s, it could not be solved except for the flow in simple geometries (*e.g.*, straight pipe). In 1904, Ludwig Prandtl (1875-1953) first proposed the boundary layer approximation; in his idea, the flow is divided into 2 regions (Figure 1):

- **outer flow region**: flow can be approximated as *inviscid* and *irrotational*; the velocity field in this region is solvable using the continuity equation and Euler equation (simplified from N-S equation for inviscid fluid flow), and the pressure field is solved using Bernoulli's theorem.
- **inner flow region**: flow near the wall, where viscous effects and rotationality cannot be neglected. We need to solve the boundary layer equation.

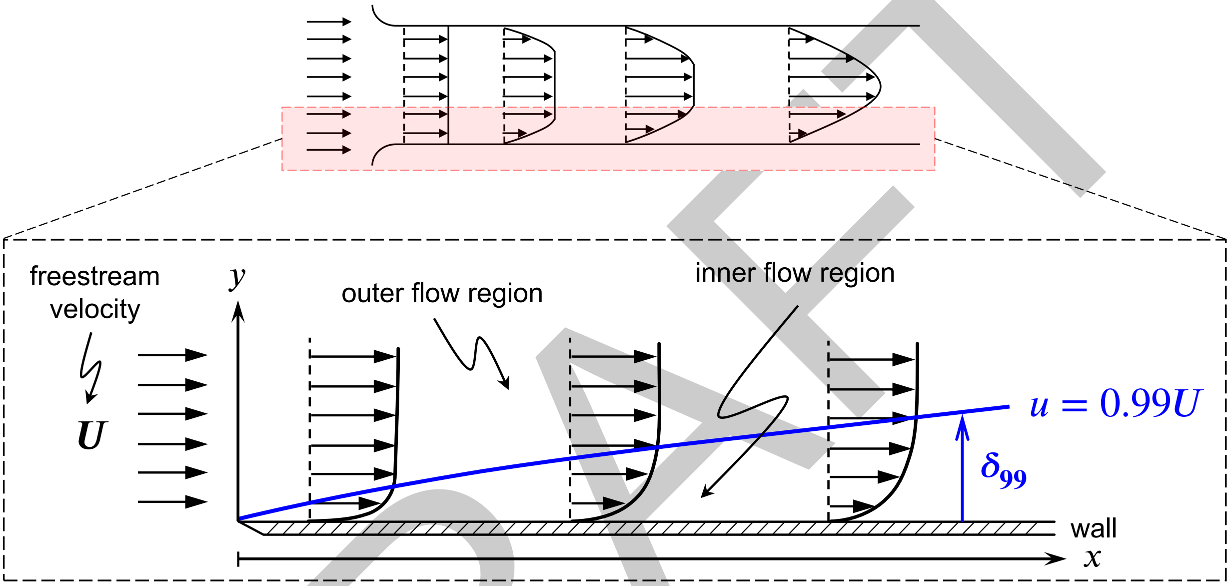

FIG. 1: A flat plate parallel to an oncoming flow. The near wall region is the boundary layer, where viscous effects affect the flow. δ_{99} denotes the boundary layer thickness where $u = 99\%U$. Also, note that δ_{99} is NOT a streamline!

Boundary Layer Equation The boundary layer equation is an approximation to the N-S equation. To derive such, we need to non-dimensionalise the x-component of the N-S momentum equation. Starting by defining the non-dimensional variables

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{\delta}, \qquad u^* = \frac{u}{U}, \qquad v^* = \frac{v}{V}, \qquad p^* = \frac{p}{P_0} = \frac{p}{\rho U^2}$$

where L is the characteristic length scale, δ is the thickness of the boundary layer, U, V are the velocity scales in the x- and y-directions, respectively. $P_0 = \rho U^2$ is the characteristic pressure, derived from the Bernoulli's theorem.

---

1. The non-dimensional continuity equation is

$$\frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{V}{\delta}\frac{\partial u^*}{\partial y^*} = 0. \tag{1}$$

Note that, to satisfy the non-dimensional continuity equation, the order of magnitude of the first term must be balanced to that of the second term, *i.e.*, $\frac{U}{L}$ and $\frac{V}{\delta}$ should be of the same order of magnitude:

$$\mathcal{O}\left(\frac{U}{L}\right) + \mathcal{O}\left(\frac{V}{\delta}\right) = 0, \quad \Rightarrow \quad \frac{U}{L} \sim \frac{V}{\delta} \quad \Rightarrow \quad V \sim \frac{U\delta}{L} \tag{2}$$

2. The non-dimensional x-momentum equation is

$$\frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{UV}{\delta}v^*\frac{\partial u^*}{\partial y^*} = -\frac{U^2}{L}\frac{\partial p^*}{\partial x^*} + \nu\frac{U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right). \tag{3}$$

To further simplify this equation, we can take a few actions

- Use the relation derived from Equation 2 to eliminate V from Equation 3, *i.e.*, $\frac{UV}{\delta} = \frac{U}{\delta} \cdot \frac{U\delta}{L} = \frac{U^2}{L}$;
- Multiply Equation 3 by the term L/U^2.

So far, the non-dimensional x-momentum equation looks like

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{L^2}{\delta^2} \frac{\partial^2 u^*}{\partial y^{*2}} \right). \tag{4}$$

Further,

- We restrict the analysis to 'narrow' channels only: $L/\delta \gg 1$.
- We are interested in the type of flow that $\mathrm{Re} \gg 1$. This ensures that $1/\mathrm{Re}$ term is safe to be eliminated.

So far, the revised non-dimensional x-momentum equation looks like

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2} \frac{\partial^2 u^*}{\partial y^{*2}}. \tag{5}$$

The last question regards the term $\frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2}$, since $1/\mathrm{Re} \ll 1$ but $L/\delta \gg 1$, which term dominates? We know the order of magnitude of the L.H.S. and the R.H.S. of Equation 5 must balance:

$$\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2} \right),$$

Obviously, $\mathcal{O}\left(\frac{1}{\mathrm{Re}} \frac{L^2}{\delta^2} \right) = \mathcal{O}(1)$. This means, $\frac{\delta}{L} \sim \mathrm{Re}^{-1/2}$.

3. Similarly, the non-dimensional y-momentum equation can be simplified as

$$\frac{\partial p^*}{\partial y^*} = 0. \tag{6}$$

Re-dimentionalise Equation 1, Equation 5, and Equation 6, which are the boundary layer equations:

(mass)	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$	(7)
(x-momentum)	$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$	(8)
(y-momentum)	$\frac{\partial p}{\partial y} = 0.$	(9)

Boundary Conditions For the type of the flow as illustrated in Figure 1, the boundary conditions are

$$u = U, \quad \text{at } x = y = 0$$
$$u = v = 0, \quad \text{at } y = 0, x \neq 0$$
$$u = U, \quad \text{as } y \to \infty$$

Displacement Thickness The boundary layer thickness, δ_{99} can be difficult to measure directly. One alternative approach is finding the equivalence of δ_{99} with the **displacement thickness**, $\delta_1(x)$. As illustrated by Figure 2(a), $\delta_1(x)$ is a thin plate that *obstructs* the inviscid flow (stagnant layer).

The expression of $\delta_1(x)$ is derived by equating the total mass flow at the inlet and at the inviscid (unobstructed) region,

$$\rho \int_0^\infty u(x,y)\mathrm{d}y = \rho \int_{\delta_1}^\infty U\mathrm{d}y.$$

Divide both sides by ρU, and split the integral,

$$\rho \int_0^\infty u^* \mathrm{d}y = \int_{\delta_1}^\infty \mathrm{d}y \implies \int_0^\infty u^* \mathrm{d}y = \int_0^\infty \mathrm{d}y - \int_0^{\delta_1} \mathrm{d}y \implies \boxed{\delta_1(x) = \int_0^\infty (1 - u^*)\mathrm{d}y}.$$

Momentum Thickness The momentum thickness, $\delta_2(x)$, is an alternative approximation of the boundary layer thickness, for which $\delta_2(x)$ has the same momentum deficit as the actual boundary layer profile, as shown by Figure 2(b).

Equating the 'artificial' momentum deficit created by δ_2 to the real momentum deficit raised from the velocity deficit, we have

$$\underbrace{\rho \int_0^{\delta_2} U^2 \, \mathrm{d}y}_{\text{momentum deficit by } \delta_2} = \int_0^{\infty} \rho u \cdot \underbrace{(U - u)}_{\text{velocity deficit}} \, \mathrm{d}y \implies \boxed{\delta_2(x) = \int_0^{\infty} u^*(1 - u^*) \, \mathrm{d}y}.$$

Despite the abstraction lies in the concept of momentum thickness, it is particularly useful in finding the fluid drag and skin friction on the plate.

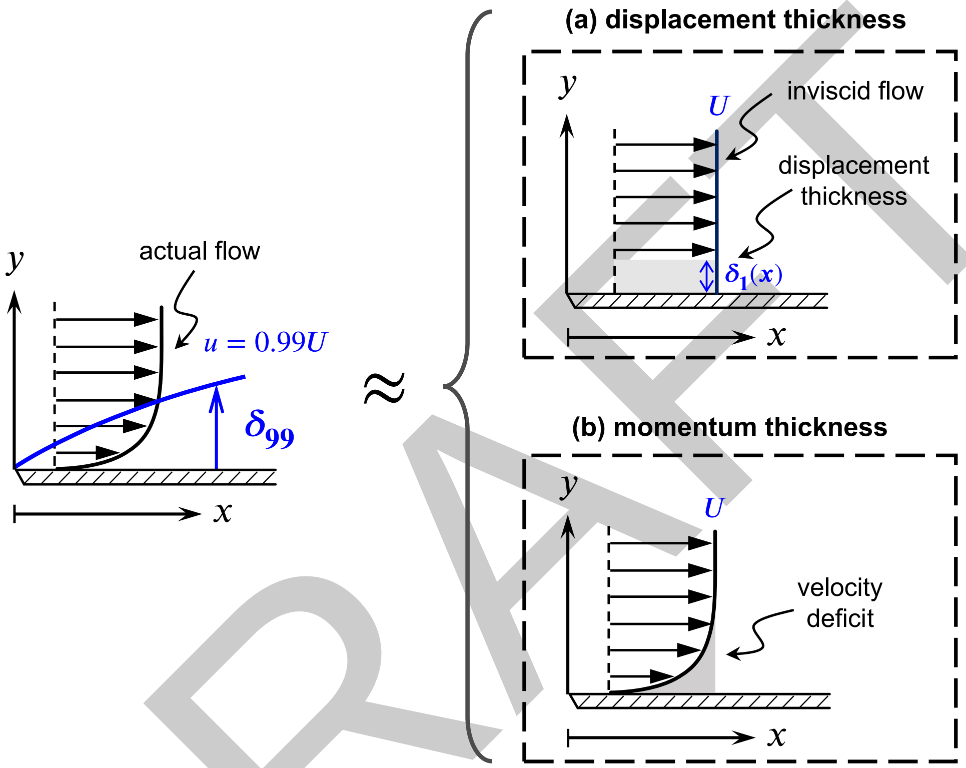

FIG. 2: Two approximations of the thickness of an actual boundary layer: (a) displacement thickness and (b) momentum thickness.