3.1 Fluid Viscosity

For the Newtonian fluid, the dynamic viscosity μ $[\mathrm{Pa \cdot s}]$ is a fixed constant; whereas for the non-Newtonian fluid, the viscosity varies with the shear stress τ $[\mathrm{Pa}]$ and shear rate $\dot{\gamma}$ $[1/\mathrm{s}]$.

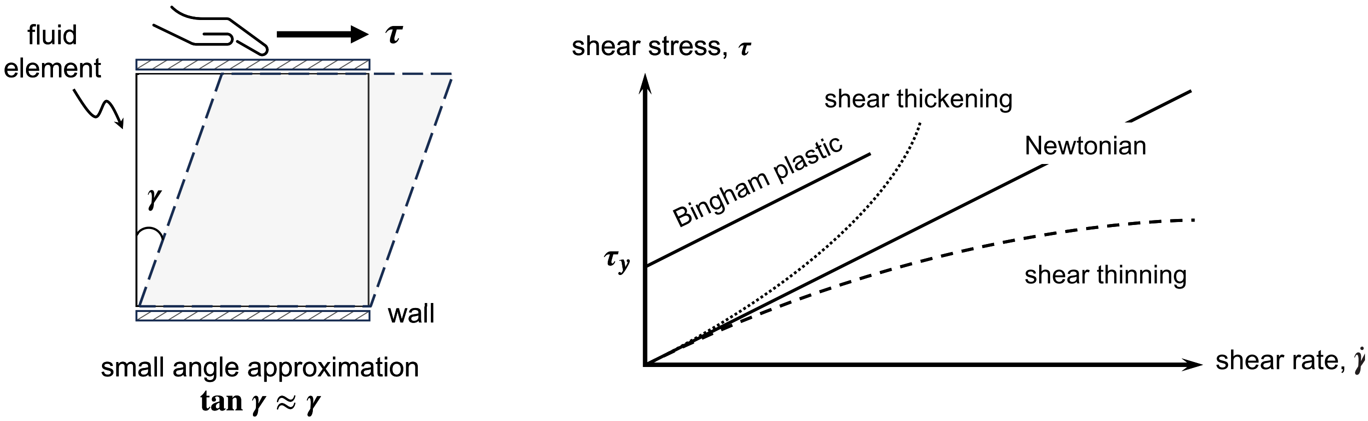

FIG. 1: Left: the concept of shear strain γ in a simple shear flow; Right: the rheological behaviour of viscous fluids can be classified by the shear stress - shear rate ($\dot{\gamma} = \mathrm{d}\gamma/\mathrm{d}t$) relations.

- Shear thickening: μ increases with shear rate - *e.g.*, cornstarch paste;
- Shear thinning: μ decreases with shear rate - *e.g.*, ketchup, blood;
- Bingham plastic: a yield stress τ_y impedes the fluid flow until $\tau > \tau_y$.

Although the blood is frequently modelled as a Newtonian fluid, it is shear thinning with yield (*a.k.a.* Bingham pseudoplastic). The non-Newtonian behaviours of blood are due to the cell suspension (rather than the plasma), hence, the viscosity is Hematocrit-dependent.

3.2 Flow in a Rectangular Duct

Consider the flow in a rectangular duct (length L, width w, height h) in the Cartesian coordinate system (Figure 2).

Assumptions

- Fluid is homogeneous, incompressible and Newtonian with viscosity μ and density ρ;
- Flow has reached the steady state: $\partial \mathbf{u}/\partial t = 0$;
- Flow is fully developed along the x-direction: $\partial \mathbf{u}/\partial x = 0$;
- Zero velocity along the y- and z-directions: $v = 0$, $w = 0$;
- Negligible body force: $\mathbf{f} = 0$.

Boundary Conditions Symmetrical flow profile at $y = 0$ and $z = 0$; no-slip condition at the wall $y = \pm h/2$, $z = \pm w/2$.

Aim Analytically solve for the flow velocity in the x-direction.

Solution The x-momentum equation is reduced to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Using separation of variables[1], the analytical solution of u is

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[y^2 - \left(\frac{h}{2}\right)^2 - \sum_{n=0}^{\infty} A_n \cos\left(\frac{\lambda_n y}{h/2}\right)\cosh\left(\frac{\lambda_n z}{h/2}\right)\right], \quad \text{where } A_n = \frac{h^2(-1)^n}{\lambda_n^3 \cosh\frac{\lambda_n w}{h}}, \quad \lambda_n = \frac{(2n+1)\pi}{2}.$$

[1] for the full derivation, see the Supplementary slides posted on Blackboard

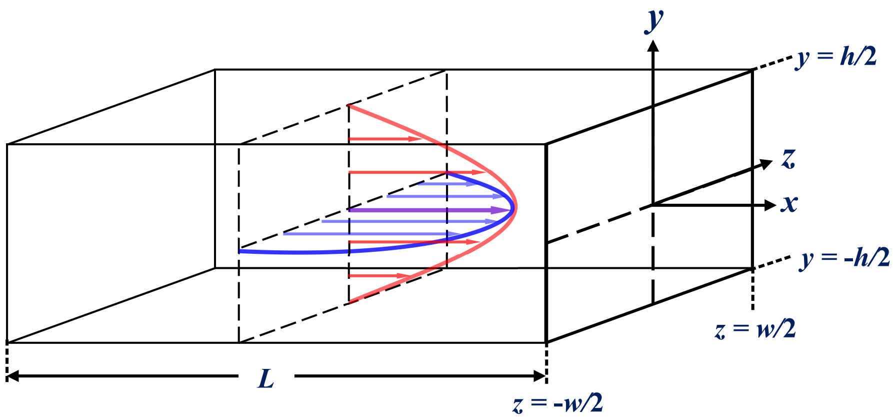

FIG. 2: The schematic for the flow in a rectangular duct.

Integrating u over the area, the flux Q can be expressed as

$$Q = \frac{\partial p}{\partial x}\frac{wh^3}{12\mu}\left[6\left(\frac{h}{w}\right)\sum_{n=0}^{\infty}\lambda_n^{-5}\tanh\left(\frac{\lambda_n w}{h}\right) - 1\right] \approx \frac{\partial p}{\partial x}\frac{wh^3}{12\mu}\left[1 - 0.6274\left(\frac{h}{w}\right)\right].$$

Finally, by $Q = \Delta p/R$, the flow resistance is

$$R = \frac{\Delta p}{Q} = \frac{12\mu L}{wh^3\left[1 - 0.6274\left(\frac{h}{w}\right)\right]}.$$

3.3 Womersley Flow

Motivation To approximate the pulsatility nature of the flow in the cardiovascular system.

Assumptions

- Fluid is homogeneous, incompressible and Newtonian with viscosity μ and density ρ;
- Flow in a long straight tube, with a perfect circular cross-section at radius a;
- Axisymmetric along the θ-axis: $\partial/\partial\theta = 0$;
- The flow is fully developed along the z-axis: $\partial \mathbf{u}/\partial z = 0$;
- No swirls: $u_\theta = 0$;
- No velocity along the radial direction: $u_r = 0$;
- Negligible body force: $\mathbf{f} = 0$.

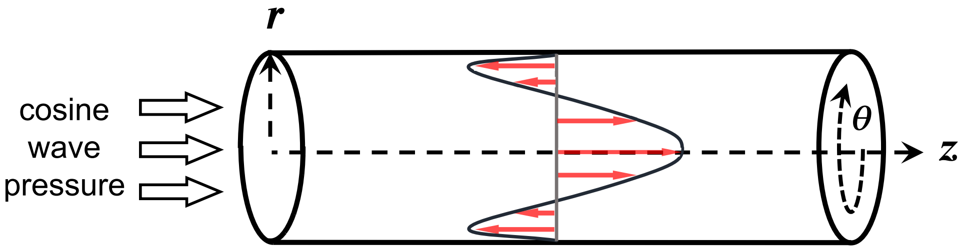

FIG. 3: The schematic of the Womersley flow in a pipe.

Boundary Conditions No-slip condition on the wall, parabolic condition as Poiseuille flow.

Solution Procedure

Step 1 The z-momentum equation

$$\rho\left(\frac{\partial u_z}{\partial t}+\cancelto{0}{u_r\frac{\partial u_z}{\partial r}}+\cancelto{0}{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}+\cancelto{0}{u_z\frac{\partial u_z}{\partial z}}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\cancelto{0}{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}}+\cancelto{0}{\frac{\partial^2 u_z}{\partial z^2}}\right]+\cancelto{0}{\rho f_z}$$

$$\Rightarrow\quad \rho\frac{\partial u_z}{\partial t}=-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right).$$

Assume the pressure gradient is sinusoidal: $\partial p/\partial z=\frac{G_0}{2}e^{i\omega t}$, and following the sinusoidal z-velocity: $u_z = U(r)e^{i\omega t}$:

$$\left[i\omega U\rho+\frac{G_0}{2}-\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)\right]e^{i\omega t}=0 \quad\xrightarrow{\text{2nd-order ODE}}\quad \frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{i\omega\rho}{\mu}U=-\frac{G_0}{2\mu}.$$

Step 2 The full solution of $U(r)$ involves a complementary function, which is formulated with the Bessel function of the 1st kind at 0th order, J_0; also the particular integral, $U_{pi}=-G_0/2i\omega\rho$:

$$U(r)=\frac{iG_0}{2\omega\rho}\left[1-\frac{J_0(i^{3/2}\alpha\frac{r}{a})}{J_0(i^{3/2}\alpha)}\right],\quad \text{with}\quad J_0(s)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!k!}\left(\frac{s}{2}\right)^{2k},$$

and α denotes the non-dimensional **Wormersley number**: $\alpha=a\sqrt{\frac{\omega\rho}{\mu}}=a\sqrt{\frac{\omega}{\nu}}$.

Step 3 To recover u_z from $U(r)$:

$$u_z(r,t)=\frac{i}{\omega\rho}\frac{\partial p}{\partial z}\left[1-\frac{J_0(i^{3/2}\alpha\frac{r}{a})}{J_0(i^{3/2}\alpha)}\right]=\frac{iG_0}{2\omega\rho}\left[1-\frac{J_0(i^{3/2}\alpha\frac{r}{a})}{J_0(i^{3/2}\alpha)}\right]e^{i\omega t}$$

Ostensibly, this solution is defined in the complex domain; but for simplicity, we only consider the real part to interpret its physical meaning.

Extended Properties

1. Wall shear stress:

$$\tau_{rz}=\mu\frac{\partial u_z}{\partial r}=\mu\Re\left\{-\frac{a}{i^{3/2}\alpha}\left(\frac{J_1(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}\right)\frac{\partial p}{\partial z}\right\},\quad \text{with}\quad J_1(s)=-\frac{\partial J_0(s)}{\partial s}.$$

2. Volume flow rate:

$$Q(t)=\int_0^a 2\pi r u_z \mathrm{d}r=\Re\left\{-\frac{\pi a^4}{i\mu\alpha^2}\left(1-\frac{2J_1(i^{3/2}\alpha)}{\alpha i^{3/2}J_0(i^{3/2}\alpha)}\right)\frac{\partial p}{\partial z}\right\}.$$

The Wormersley Number The Wormersley number α is the ratio between unsteady inertia force and viscous force.

- $\alpha \le 1$: **Quasi-steady**, the velocity profile is basically scaled Poiseuille flow, mainly observed in the microvasculatures (*e.g.*, capillaries, venules);
- $\alpha > 1$: **Oscillatory**, the velocity profile is balanced between viscous forces at the wall and inertial forces in the centre. Common in large arteries (*e.g.*, ascending aorta, carotid artery).

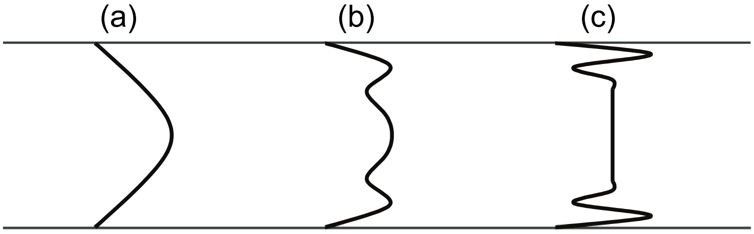

FIG. 4: Womersley flow profiles. (a) Low α (viscous dominates), (b) intermediate α, (c) high α (inertia dominates).