5.1 Dimensional Analysis

Buckingham-Π Theorem The Buckingham-Π theorem states that if an equation involving k variables is dimensionally homogeneous (*i.e.*, L.H.S. units = R.H.S. units),

$$u_1 = f(u_2, u_3, ..., u_k),$$

it can be reduced to a relationship among $(k-r)$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables,

$$\Pi_1 = \phi(\Pi_2, \Pi_3, ...\Pi_{k-r}).$$

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, L; Density, ρ; Frequency of oscillating flow, ω; Pressure, p; Speed of sound, c; Surface tension, σ_s ; Velocity, U.

Dimensionless group	Name	Interpretation	Types of Applications
$\rho U L/\mu$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$U/\sqrt{gL}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$p/\rho U$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
U/c	Mach number, Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\omega L/U$	Strouhal number, St	$\frac{\text{inertia(local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\rho U^2 L/\sigma_s$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

Table 1: Common variables and dimensionless groups in fluid mechanics.

Parameter	Symbol	Dimensions	Parameter	Symbol	Dimensions
Acceleration	a	$[L^1T^{-2}]$	Surface tension	σ_s	$[M^1T^{-2}]$
Angle	θ, ϕ, etc.	1 (none)	Velocity	U	$[L^1T^{-1}]$
Density	ρ	$[M^1L^{-3}]$	Viscosity	μ	$[M^1L^{-1}T^{-1}]$
Force	F	$[M^1L^1T^{-2}]$	Volume flow rate	Q	$[L^3T^{-1}]$
Frequency	f	$[T^{-1}]$	Pressure	p	$[M^1L^{-1}T^{-2}]$

Table 2: Table of parameters with symbols and primary dimensions in two columns. $[M]$: mass, $[T]$: time; $[L]$: length.

5.2 Non-Dimensional Navier-Stokes Equation

- Define the non-dimensional variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{L}, \qquad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \qquad t^* = \frac{t}{L/U}, \qquad p^* = \frac{p}{P_0},$$

  where L, U are the characteristic length and velocity, respectively.

- The dimensionless Navier-Stokes momentum equation is

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\right) = -\frac{P_0}{\frac{\mu U}{L}}\nabla^* p^* + \nabla^{*2}\mathbf{u}^*,$$

  where $P_0 = \frac{\mu U}{L}\max(1, \mathrm{Re})$, *i.e.*, the viscous scale $(\mathrm{Re} < 1)$ or dynamic scale $(\mathrm{Re} > 1)$. This formulation ensures the pressure term has the same order of magnitude as other terms, since there is no natural scaling for pressure.

- The dimensionless continuity equation is

$$\nabla^* \cdot \mathbf{u}^* = 0.$$

Small Re **flow (**$\mathrm{Re} \ll 1$**)** $P_0 = \mu U/L$ and the L.H.S. eliminated,

$$\cancelto{0}{\mathrm{Re}\Big(\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^*\Big)} = -\nabla^* p^* + \nabla^{*2}\mathbf{u}^* \quad \Longrightarrow \quad \nabla^* p^* = \nabla^{*2}\mathbf{u}^* \quad \Longleftrightarrow \quad \mu\nabla^2\mathbf{u} = \nabla p$$

which is known as the **Stokes equation** that can be solved analytically due to its linearity.

Governing Equation of Stokes Flow

Define the vorticity as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$

$$\mu\nabla^2\mathbf{u} = -\mu\nabla\times\boldsymbol{\omega} \qquad \text{due to} \qquad \nabla\times\boldsymbol{\omega} = \nabla\times(\nabla\times\mathbf{u}) = \cancel{\nabla\cdot\mathbf{u}} - \nabla^2\mathbf{u}.$$

Further, take the curl of $\mu\nabla^2\mathbf{u} = \nabla p$:

$$\underbrace{\nabla\times\nabla p}_{\text{“curl of grad is zero”}} = \nabla\times(\mu\nabla^2\mathbf{u}) \quad \Longrightarrow \quad \begin{aligned} 0 &= -\mu\nabla\times(\nabla\times\boldsymbol{\omega}) \\ 0 &= -\mu[\underbrace{\nabla(\nabla\cdot\boldsymbol{\omega}) - \nabla^2\boldsymbol{\omega}}_{\text{by } \nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}}] \\ 0 &= -\mu[\nabla(\underbrace{\nabla\cdot\nabla\times\mathbf{u}}_{\text{“div of curl is zero”}}) - \nabla^2\boldsymbol{\omega}]. \end{aligned}$$

The above derivation results in $\nabla^2\boldsymbol{\omega} = 0$, which is the governing equation of the Stokes flow.

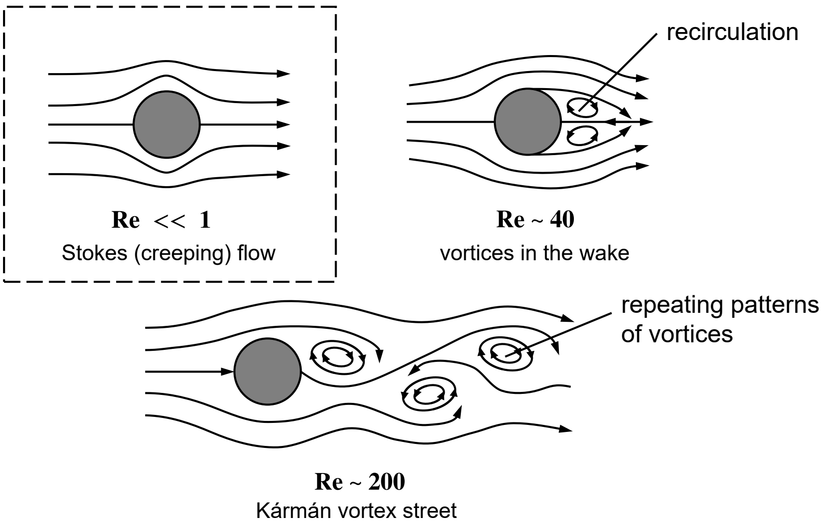

FIG. 1: Flow passing around a cylinder at different Reynolds numbers. The top left scenario depicts the Stokes flow when $\mathrm{Re} \ll 1$ - no flow separation.

Large Re **flow (**$\mathrm{Re} \gg 1$**)** $P_0 = \rho U^2$ and the viscus term eliminated (hence, the fluid is approximated nearly inviscid),

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* \cdot \nabla^*)\mathbf{u}^* = -\nabla^* p^* \quad \Longrightarrow \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p,$$

which is known as the **Euler equation**.

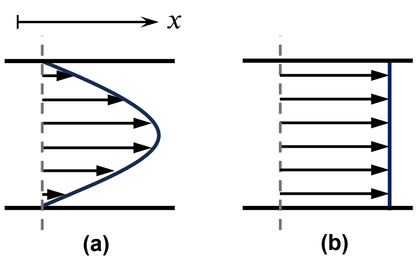

FIG. 2: The velocity profile of flow between two parallel plates when the fluid is (a) affected by viscosity, (b) inviscid.

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