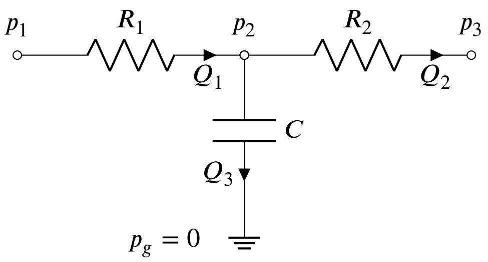
7.1 Lumped Parameter Modelling

Resistance, Compliance, and Intertance

Resistance	Compliance	Inertance
$Q = \Delta p/R$	$Q = C\dfrac{\partial p}{\partial t}$	$p = L\dfrac{\partial Q}{\partial t}$

- **Resistance** R: analogous to the electrical resistance which models the dissipation of energy. The mass flow rate Q is analogous to the electrical current (usually denoted by I), and the pressure p is analogous to the electrical voltage (usually denoted by V).
- **Compliance** C: this models the expansion of cardiovascular chambers under pressure, allowing it to store more fluid.
- **Inductor** L: this models the inertance of the fluid. When the fluid momentum is substantial, as the pressure on forward-flowing fluid reverses, the fluid will not suddenly reverse its direction, but decelerate over a transient.

Solving a Lumped Parameter Network Consider the example lumped parameter network,



... which yields a linear system with 4 unknowns (p_2, Q_1, Q_2, Q_3) and 4 simultaneous equations:

$$\begin{cases} p_2 - p_1 = R_1 Q_1, \\ p_3 - p_2 = R_2 Q_2, \\ Q_3 = C(p_2^{(t)} - p_2^{(t-1)})/\Delta t, \\ Q_1 = Q_2 + Q_3. \end{cases}$$

Note that $p_2^{(t-1)}$ denotes the pressure p_2 at the previous time step $t-1$; $(p_2^{(t)} - p_2^{(t-1)})/\Delta t$ is an expression of the time derivative in the backward Euler fashion. (*cf.* electrical capacitor $I = C \cdot \mathrm{d}V/\mathrm{d}t$).

The above linear system can be arranged into a matrix system, $\mathbf{Ax} = \mathbf{b}$,

$$\begin{bmatrix} 1 & -R_1 & 0 & 0 \\ -1 & 0 & -R_2 & 0 \\ -1 & 0 & 0 & \frac{\Delta t}{C} \\ 0 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p_2 \\ Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ -p_3 \\ -p_2^{(t-1)} \\ 0 \end{bmatrix},$$

and can be easily solved by inversion of the coefficient matrix: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

7.2 Windkessel Models

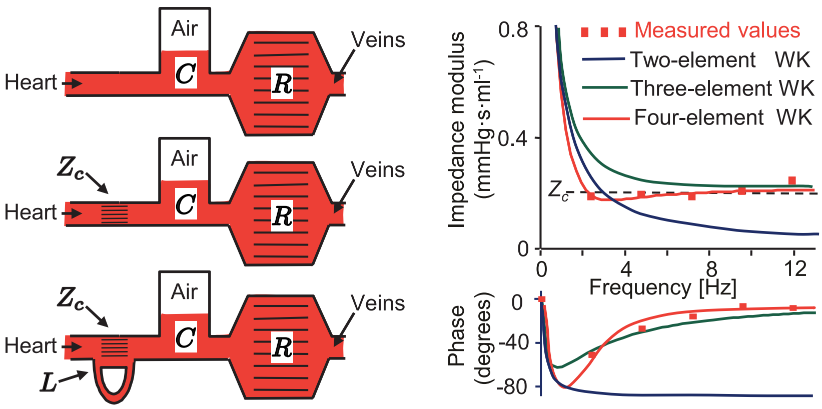
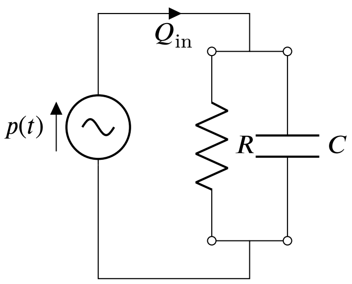


FIG. 1: Left: the mechanical equivalence of three Windkessel models; Right: Input impedances of the three Windkessels compared with the measured input impedance. (Westerhof *et. al*)

2-element Windkessel Model

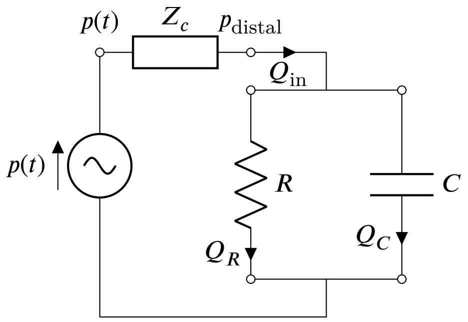


Governing Equation:

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} + \frac{p(t)}{RC} = \frac{Q_{\text{in}}}{C}$$

where C denotes the vessel compliance (elasticity), R denotes the peripheral (distal) resistance.

3-element Windkessel Model



Governing Equation:

$$\frac{\partial p(t)}{\partial t} + \frac{p(t)}{RC} = \frac{Q_{in}}{C}\left(1 + \frac{Z_c}{R}\right) + Z_c \frac{\partial Q_{\text{in}}}{\partial t}$$

where Z_c is the characteristic impedance, $p(t) - p_{\text{distal}} = Z_c Q_{\text{in}}$.

Derivation

Apply Kirchhoff's Current Law at node p_{distal}: $Q_{\text{in}} = Q_R + Q_C$. Moreover, since $p(t) - p_{\text{distal}} = Z_c Q_{\text{in}} \Rightarrow p_{\text{distal}} = p(t) - Z_c Q_{\text{in}}$.

- Current passes through the resistor R:

$$Q_R = \frac{p_{\text{distal}}}{R} = \frac{p(t) - Z_c Q_{\text{in}}}{R} = \frac{p(t)}{R} - \frac{Z_c Q_{\text{in}}}{R}.$$

- Current passes through the capacitor C:

$$Q_C = C\frac{\partial p_{\text{distal}}}{\partial t} = C\frac{\partial[p(t) - Z_c Q_{\text{in}}]}{\partial t} = C\frac{\partial p(t)}{\partial t} - C Z_c \frac{\partial Q_{\text{in}}}{\partial t}.$$

Hence, the total flow Q_{in} is

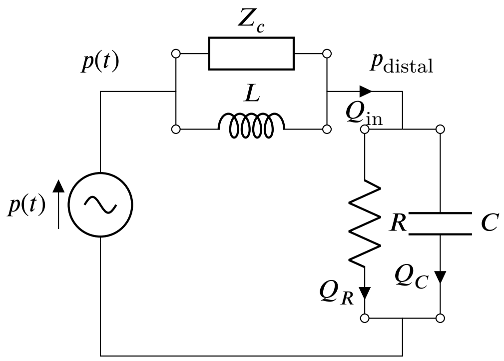
$$\begin{aligned} Q_{\text{in}} &= Q_R + Q_C \\ &= \frac{p(t)}{R} - \frac{Z_c Q_{\text{in}}}{R} + C\frac{\partial p(t)}{\partial t} - C Z_c \frac{\partial Q_{\text{in}}}{\partial t}, \end{aligned}$$

rearrange, we get

$$C\frac{\partial p(t)}{\partial t} + \frac{p(t)}{R} = \left(1 + \frac{Z_c}{R}\right) Q_{\text{in}} + C Z_c \frac{\partial Q_{\text{in}}}{\partial t}.$$

Divide both sides of the equation above by C, we will get the final governing equation as presented.

4-element Windkessel Model



Governing Equation:

$$\frac{\partial p}{\partial t} + \frac{p(t)}{RC} = \frac{Q}{C}\left(1 + \frac{Z_{\text{total}}}{R}\right) + Z_{\text{total}} \frac{\partial Q}{\partial t}$$

where $Z_{\text{total}} = \dfrac{j\omega L Z_c}{j\omega L + Z_c}$ is the total impedance of the parallel network - the characteristic impedance, Z_c and the inductor, L.

Derivation

Apply Kirchhoff's Current Law at node p_{distal}: $Q_{\text{in}} = Q_R + Q_C$. However, we need to express p_{distal} in terms of $p(t)$, hence need to solve the total impedance of the Z_c-L parallel network:

$$\frac{1}{Z_{\text{total}}} = \frac{1}{Z_c} + \frac{1}{j2\pi f L} = \frac{j2\pi f L + Z_c}{j2\pi f L Z_c} \implies Z_{\text{total}} = \frac{j2\pi f L Z_c}{j2\pi f L + Z_c}.$$

Note that sometimes $2\pi f$ is denoted as ω, which is the angular frequency. Now, $p(t) - p_{\text{distal}} = Z_{\text{total}} Q_{\text{in}}$. The rest of this derivation follows the same procedure for 3-WK.

Necessity of the inductance in 4-WK? Better capture the frequency characteristics of the flow.

- At the low f range: $2\pi f L \ll Z_c$, hence $Z_{\text{total}} \to 0$, which removes the characteristic impedance in the whole circuit;
- At the high f range: $2\pi f L \gg Z_c$, hence $Z_{\text{total}} \to Z_c$.

This means the inductance has no effect when the flow is steady, providing a zero resistance pathway to the rest of the circuit under steady flow conditions.

7.3 Moens-Korteweg Model of Pulse Wave Velocity

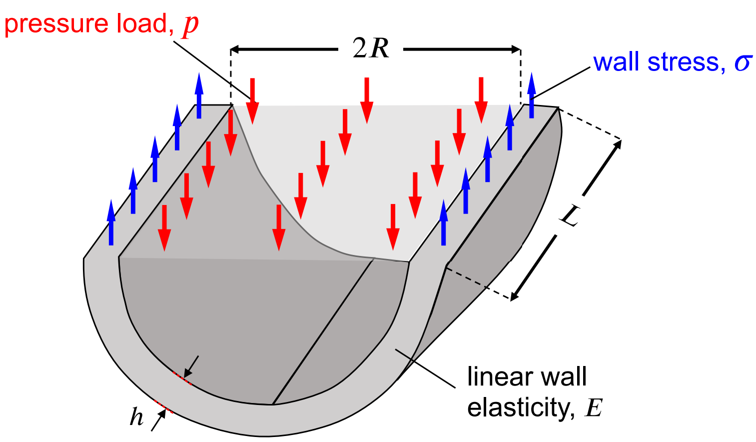


FIG. 2: The schematic for the derivation of Moens-Korteweg equation.

Equation 1 Assume linear elasticity (fixed Young's modulus, E), the stress(σ)-strain(ε) relation is

$$\sigma = E\varepsilon = E\frac{\Delta R}{R} \quad \text{with} \quad \varepsilon = \frac{(2\pi(R+\Delta R) - 2\pi R)}{2\pi R} = \frac{\Delta R}{R}.$$

Applying Newton's 2$^{\text{nd}}$ Law and re-arranging the expression leads to an expression of the pressure,

$$m_{\text{wall}} a_{\text{wall}} = F_{\text{pressure}} - F_{\text{wall}}$$
$$0 = 2RL \times P - 2Lh \times \sigma \quad \Rightarrow \quad P = \frac{\sigma h}{R} = \frac{Eh}{R^2}\Delta R.$$

Differentiating p w.r.t. t, this leads to **equation 1**,

$$\boxed{\frac{\partial p}{\partial t} = \frac{Eh}{R^2}\frac{\partial \Delta R}{\partial t}}.$$

Equation 2 Integrating the continuity equation over the vascular cross-sectional area

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \xcancel{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}^{\,0,\ \text{axis-symmetrical}} + \frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad \int \left(\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right)\partial A = 0$$

$$\Rightarrow \quad \int_{r=0}^{r=R}\left(\frac{1}{r}\frac{\partial r u_r}{\partial r}\right)2\pi r \partial r + \pi R^2 \frac{\partial \overline{u_z}}{\partial z} = 0$$

$$\Rightarrow \quad 2\pi R u_R + \pi R^2 \frac{\partial \overline{u_z}}{\partial z} = 0.$$

Re-arrange leads to the **equation 2**,

$$\boxed{u_r = -\frac{R}{2}\frac{\partial \overline{u_z}}{\partial z}},$$

where the notation $\overline{u_z}$ denotes the average z-velocity across cross-section.

Equation 3 Assume negligible convective acceleration and no viscous losses, the Navier-Stokes z-momentum equation can be simplified as,

$$\rho\left(\frac{\partial u_z}{\partial t} + \xcancel{u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}}^{\,0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\xcancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}}^{\,0}\right] + \xcancel{\rho f_z}^{\,0} \Rightarrow \boxed{\rho\frac{\partial \overline{u_z}}{\partial t} = -\frac{\partial p}{\partial z}}.$$

Derivation of PVW First, let $u_r = \dfrac{\partial \Delta R}{\partial t}$, this equates **equation 1** and **equation 2** and leads to **equation 4**

$$\underbrace{u_r = -\frac{R}{2}\frac{\partial \overline{u_z}}{\partial z}}_{\text{equation 2}} = \underbrace{\frac{\partial \Delta R}{\partial t} = \frac{R^2}{Eh}\frac{\partial p}{\partial t}}_{\text{equation 1}}, \quad \Rightarrow \quad \underbrace{\frac{\partial \overline{u_z}}{\partial z} = -\frac{2R}{Eh}\frac{\partial p}{\partial t}}_{\text{equation 4}}$$

Next, differentiate **equation 3** and **equation 4** w.r.t. t,

$$\rho\frac{\partial \overline{u_z}}{\partial t} = -\frac{\partial p}{\partial z} \quad \xrightarrow[\text{w.r.t. } t]{\text{differentiate}} \quad \rho\frac{\partial^2 \overline{u_z}}{\partial t \partial z} = -\frac{\partial^2 p}{\partial z^2},$$

$$\frac{\partial \overline{u_z}}{\partial z} = -\frac{2R}{Eh}\frac{\partial p}{\partial t} \quad \xrightarrow[\text{w.r.t. } t]{\text{differentiate}} \quad \frac{\partial^2 \overline{u_z}}{\partial z \partial t} = -\frac{2R}{Eh}\frac{\partial^2 p}{\partial t^2},$$

which allows us to equate the R.H.S. as

$$\frac{\partial^2 p}{\partial z^2} = \frac{2R\rho}{Eh}\frac{\partial^2 p}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 p}{\partial t^2} = \underbrace{\frac{Eh}{2R\rho}}_{c^2}\frac{\partial^2 p}{\partial z^2},$$

which can be subsequently re-arranged as the wave equation. Denote the term $\dfrac{Eh}{2R\rho} = c^2$, for which the term c is the expression of the wave speed of pressure (*a.k.a.* pulse wave velocity, PVW). By definition, PVW increases with the stiffness of the vessels and decreases with the radius of the vessel.