Poisson and Laplace Equations in Electrostatic Fields

In electromagnetism, Gauss's Law states that

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon \tag{1}$$

where **E** denotes the electric field, ρ is the electric charge density, and ε is the permittivity of the medium. Further, the electric field is related to the electric potential V:

$$\mathbf{E} = -\nabla V. \tag{2}$$

Combining Equation 1 and 2 together:

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = \boxed{-\nabla^2 V = \frac{\rho}{\varepsilon}} \tag{3}$$

The boxed equation is known as the **Poisson Equation** of electrostatic fields.

If the electric charge density is 0, then

$$\nabla^2 V = 0. \tag{4}$$

This equation is known as **Laplace Equation** of electrostatic fields.

Example: Solution to Laplace Equation by Separation of Variables

In one hospital, patients who underwent pacemaker implantations are wheeled through a long corridor. After a new lighting system was installed on the roof, unexpected pacemaker failures were reported.

As a bioengineer, you are taking charge of the investigation. You suspect the pacemakers were failing due to an excessively high electric field in the corridor; therefore, you carried out a few measurements.

In terms of the dimension, the corridor has a rectangular cross-section, as shown in Figure 1: $0 \le x \le a$ and $0 \le y \le b$, where x is the horizontal direction (wall to wall, width) and y is vertical (ground to roof, height). The corridor is straight and sufficiently long.

You also measured the electrical potential difference between the walls, ground and roof. The measurements read

- No potential difference between the 2 walls and the ground, *i.e.*
  - $V = 0$ volts at $x = 0$;
  - $V = 0$ volts at $y = 0$;
  - $V = 0$ volts at $x = a$;
- Potential difference between the ground and the roof is $V(x) = V_0$, *i.e.*
  - $V = V(x)$ volts at $y = b$.

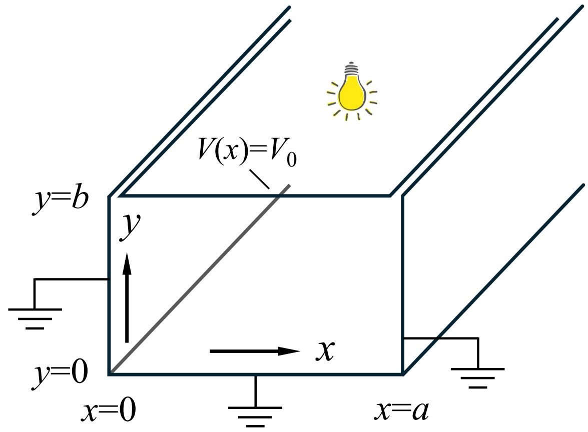

Figure 1: Sketch of the hospital corridor.

The next task is to calculate the potential in this corridor. This requires solving the Laplace equation.

Starting with Laplace's equation:

$$\nabla^2 V = 0 \quad \rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \cancelto{0}{\frac{\partial^2 V}{\partial z^2}} = 0$$

Here, we cancelled the z-direction term, as the corridor is straight and sufficiently long; by assumption, there is no variation of the electric potential in the z-direction.

To solve this 2$^{\text{nd}}$-order partial differential equation, we employ the method of **separation of variables**. Using the relation solution: $V(x, y) = X(x)Y(y)$:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 X}{\partial x^2} y + \frac{\partial^2 Y}{\partial y^2} x = 0$$

Divide both sides by xy:

$$\frac{1}{x}\frac{\partial^2 X}{\partial x^2} + \frac{1}{y}\frac{\partial^2 Y}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{1}{x}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{y}\frac{\partial^2 Y}{\partial y^2} = -k^2$$

where $-k^2$ is a constant term, and it is commonly referred to as the *separation constant*. By employing this method, the Laplace equation has been separated into two homogeneous ordinary differential equations (ODEs):

$$\frac{\partial^2 X}{\partial x^2} + k^2 x = 0 \quad \text{and} \quad \frac{\partial^2 Y}{\partial y^2} - k^2 y = 0.$$

To solve the x-dependent ODE: the characteristic equation $r^2 - 4k^2 r = 0$, there exist two complex roots of r, hence, we conclude the general solution must be in the form

$$x = A_1 e^{jkx} + A_2 e^{-jkx},$$

where j denotes the imaginary unit, A_1 and A_2 are unknown constants subject to the boundary conditions.

To solve the y-dependent ODE: the characteristic equation $r^2 + 4k^2r = 0$, there exist two real roots of r, hence, we conclude the general solution must be in the form

$$y = B_1 e^{ky} + B_2 e^{-ky}$$

where B_1 and B_2 are unknown constants subject to the boundary conditions.

Therefore,

$$V(x, y) = X(x)Y(y) = (A_1 e^{jkx} + A_2 e^{-jkx})(B_1 e^{ky} + B_2 e^{-ky})$$

Substitute 4 boundary conditions into $V(x, y)$:

1. $V = 0$ when $x = 0$:

$$0 = (A_1 + A_2)(B_1 e^{ky} + B_2 e^{-ky}) \quad \rightarrow \quad A_1 = -A_2$$

   Therefore,

$$V = A_1(e^{jkx} - e^{-jkx})(B_1 e^{ky} + B_2 e^{-ky})$$

2. $V = 0$ when $y = 0$:

$$0 = A_1(e^{jkx} - e^{-jkx})(B_1 + B_2) \quad \rightarrow \quad B_1 = -B_2$$

   Therefore,

$$V = A_1 B_1 (e^{jkx} - e^{-jkx})(e^{ky} - e^{-ky})$$

3. $V = 0$ when $x = a$:

$$\begin{aligned} 0 &= A_1 B_1 \underbrace{(e^{jkx} - e^{-jkx})}_{=2j\sin(kx)} \underbrace{(e^{ky} - e^{-ky})}_{=2\sinh(ky)} \\ &= 4jA_1B_1 \ \sin(ka) \ \sinh(ky) \end{aligned}$$

   In this case, since $\sinh(ky) \neq 0$, and if $A_1 = 0$ or $B_1 = 0$, the solution will be trivial, therefore, $A_1 \neq 0$ and $B_1 \neq 0$. The only term left $\sin(ka)$ is 0. Let $k = \frac{n\pi}{a}$, where

$n = 1, 2, 3, ...$, we have:

$$\begin{aligned} V &= \underbrace{4jA_1B_1}_{=C}\ \sin\left(\frac{n\pi}{a}x\right)\ \sinh\left(\frac{n\pi}{a}y\right) \\ &= C\ \sin\left(\frac{n\pi}{a}x\right)\ \sinh\left(\frac{n\pi}{a}y\right) \\ &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right)\ \sinh\left(\frac{n\pi}{a}y\right) \end{aligned}$$

4. $V = V(x)$ when $y = b$:

$$V(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a}x\right)\ \sinh\left(\frac{n\pi}{a}b\right)$$

Let

$$D_n = C_n \sinh\left(\frac{n\pi}{a}b\right)$$

Therefore, we can obtain the Fourier series:

$$V(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{a}x\right)$$

For $V(x) = V_0$, expand D_n:

$$\begin{aligned} D_n &= \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi}{a}x\right) \mathrm{d}x \\ &= -\frac{2}{a}\left[\frac{aV_0}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right]_0^a \\ &= \frac{2V_0}{n\pi}(1-(-1)^n) \end{aligned}$$

Note that: if n takes an even number, $D_n = \frac{4V_0}{n\pi}$; n can never take odd numbers.

We can recover the expression for C_n, then find the expression for V:

$$V(x,y) = \sum_{n=1}^{\infty} \frac{2V_0}{n\pi\ \sinh(\frac{n\pi}{a}b)}\ \sin\left(\frac{n\pi}{a}x\right)\ \sinh\left(\frac{n\pi}{a}y\right)$$

As n increases, we can obtain a Fourier series that tends to have a constant value.

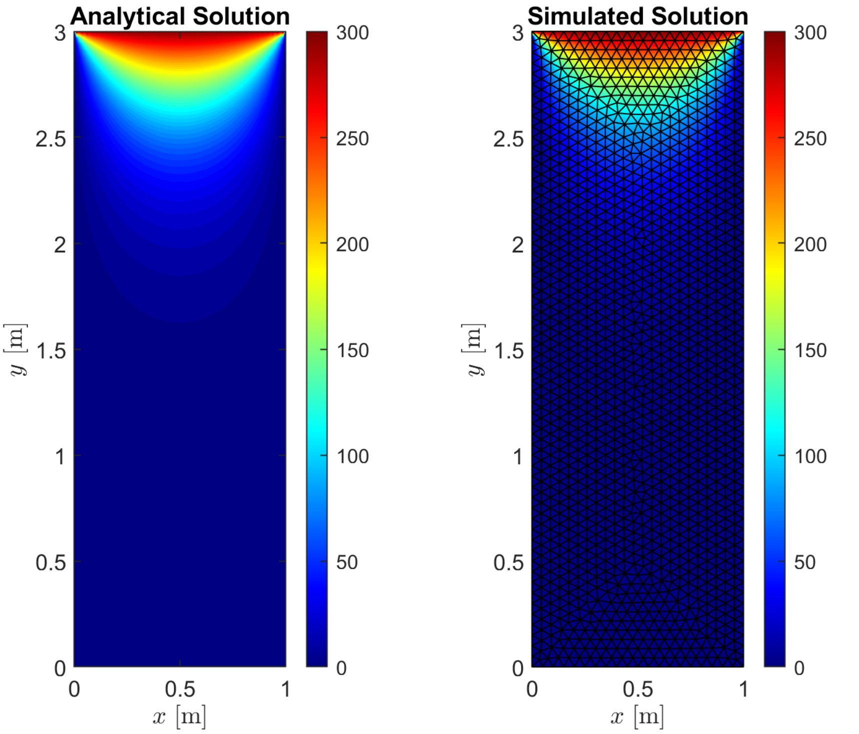

Figure 2: Analytical (left) and numerical (finite element simulation) solution plot of the electric potential field $V(x, y)$ with $a = 1$ m, $b = 3$ m, $V_0 = 300$ V, and $n = 81$.

Notes by **Binghuan Li**, last compiled on October 5, 2025. This example is adopted from the lectures of Electronics and Electromagnetics 2 at the Department of Bioengineering, Imperial College London, delivered by Martin Holloway.