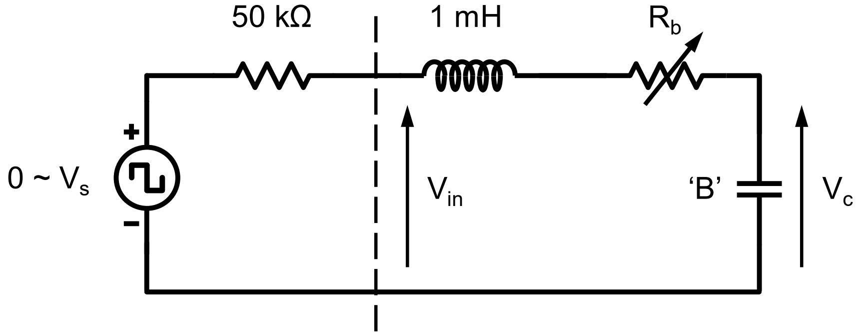
Question Statement:

Deduce (symbolic) expressions for α, ω, A and ϕ in terms of R, C, L and V_s for the circuit shown below, taking the initial charge on the capacitor when the square wave drops to zero volts to be CV_s and the current flowing at that time to be zero amps.



The governing equation for the L-C-R circuit is given as

$$L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + R\,\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{C}q = 0\ ,$$

which is a linear, homogeneous, second-order ordinary differential equation. To derive the analytical solution to this equation, assume the solution are in the following form: $q(t) = e^{\lambda t}$, then the derivatives of q are $\frac{\mathrm{d}q}{\mathrm{d}t} = \lambda e^{\lambda t}$ and $\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \lambda^2 e^{\lambda t}$, respectively. Substituting these into the governing equation:

$$(\lambda^2 L + \lambda R + 1/C)\, e^{\lambda t} = 0,$$

yielding the auxiliary equation as $(\lambda^2 L + \lambda R + q/C) = 0$. Using the quadratic equation, the solutions of λ are given by

$$\lambda_\pm = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

To this end, one needs to discuss the sign of $R^2 - \frac{4L}{C}$, it eventually affects the form of general solution we will derive.

Case 1: $R^2 - \frac{4L}{C} < 0\ \leftrightarrow R <\ 2\sqrt{\frac{L}{C}}$, there are two conjugate complex solutions of $\lambda = a \pm b\mathrm{i}$, the solution of q will be in the form

$$q(t) = k_1 e^{\lambda_+ t} + k_2 e^{\lambda_- t} =\ k_1 e^{(a+b\mathrm{i})t} + k_2 e^{(a-b\mathrm{i})t} = e^{at}\cdot(k_1 e^{\mathrm{i}bt} + k_2 e^{-\mathrm{i}bt})$$

where $a = -\frac{R}{2L}$, $b = \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}$ are the real and imaginary parts of $\lambda_\pm$; k_1, k_2 are constants yet to find. Expanding $e^{\mathrm{i}bt}$ and $e^{-\mathrm{i}bt}$ using Euler's formula:

$$e^{ibt} = \cos bt + i \sin bt$$
$$e^{-ibt} = \cos bt - i \sin bt$$

Substitute into the expression of $q(t)$ group sine terms and cosine terms:

$$\begin{aligned} q(t) &= e^{at} \cdot \left[k_1(\cos bt + i \sin bt) + k_2(\cos bt - i \sin bt)\right] \\ &= e^{at} \cdot [\underbrace{(k_1 + k_2)}_{\text{denote as } k_3} \cos bt + i\, \underbrace{(k_1 - k_2)}_{\text{denote as } k_4} \sin bt] \\ &= e^{at} \cdot \left[k_3 \cos bt + k_4 \sin bt\right]. \end{aligned}$$

Cf. the expression given out in the lab manual $q(t) = Ae^{-\alpha t} \cos(\omega t + \phi)$, we can deduce the following:

- $a = -\alpha$, hence $\alpha = \frac{R}{2L}$, which the rate of decay;
- $b = \omega = \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}$ which is the frequency of oscillation;
- k_3 and k_4 should be linked to A and ϕ; the only way to achieve this is through the formula $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$. As such, we deduce $k_3 = A \cos \phi$, $k_4 = A \sin \phi$. This is helpful, but still need to find the expression of A and ϕ using the known terms.

Apply the Dirichlet boundary condition: $q(0) = CV_s$ at $t = 0$, we know

$$A \cos \phi = CV_s$$

Apply the Neumann boundary condition: $i = \frac{dq}{dt} = 0$ at $t = 0$, we know

$$\frac{dq}{dt} = -A\, e^{-\alpha t}[\alpha \cos(\omega t + \phi) + \omega \sin(\omega t + \phi)],$$

hence,

$$\left.\frac{dq}{dt}\right|_{t=0} = -\alpha A \cos \phi - \omega A \sin \phi = -\underbrace{\frac{R}{2L}}_{\alpha}\, \underbrace{CV_s}_{A \cos \phi} - \underbrace{\frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}}_{\omega} A \sin \phi = 0$$

$$\rightarrow\ A \sin \phi = -\frac{RCV_s}{\sqrt{R^2 - \frac{4L}{C}}}$$

Divide by $A \cos \phi = CV_s$:

$$\tan \phi = -\frac{R}{\sqrt{R^2 - \frac{4L}{C}}} \quad \rightarrow \quad \phi = \operatorname{atan} -\frac{R}{\sqrt{R^2 - \frac{4L}{C}}}.$$

Having known the phase ϕ, the amplitude $A = \frac{CV_s}{\cos \phi}$.

Case 2: $R^2 - \frac{4L}{C} > 0 \leftrightarrow R > 2\sqrt{\frac{L}{C}}$, there are two real solutions of λ; , the solution of q will be in the form

$$q(t) = k_1 e^{\lambda_+ t} + k_2 e^{\lambda_- t},$$

where $\lambda_\pm = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$, k_1 and k_2 are constants subject to the boundary conditions.
Apply the Dirichlet boundary condition: $q(0) = CV_s$ at $t = 0$, we know

$$k_1 + k_2 = CV_s$$

Apply the Neumann boundary condition: $i = \frac{\mathrm{d}q}{\mathrm{d}t} = 0$ at $t = 0$, we know

$$\lambda_+ k_1 + \lambda_- k_2 = 0$$

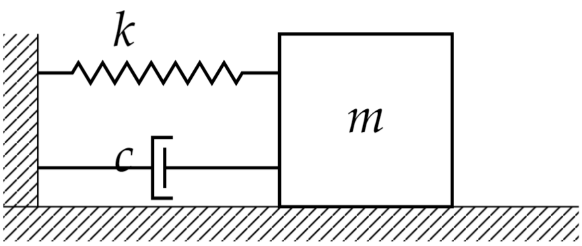
Solving the simultaneous equations giving

$$k_1 = \frac{\lambda_- CV_s}{\lambda_+ - \lambda_-} = \frac{\left(-R - \sqrt{R^2 - \frac{4L}{C}}\right) \cdot CV_s}{2\sqrt{R^2 - \frac{4L}{C}}},$$

$$k_2 = \frac{\lambda_+ CV_s}{\lambda_+ - \lambda_-} = \frac{\left(-R + \sqrt{R^2 - \frac{4L}{C}}\right) \cdot CV_s}{2\sqrt{R^2 - \frac{4L}{C}}}.$$

Case 3: $R^2 - \frac{4L}{C} = 0$, derivation is neglected here as not required in the manual.

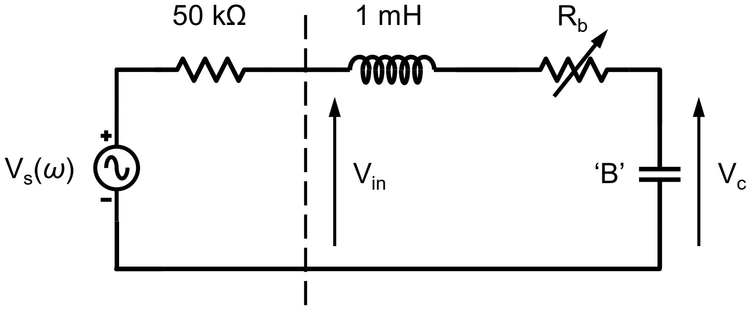
Mechanical equivalence:



The same L-C-R circuit with the capacitor initially fully charged.

Question Statement:

By $\frac{V_c}{V_{in}} = \frac{1}{(1-\omega^2 LC)+j\omega CR}$, $\left|\frac{V_c}{V_{in}}\right|$ has a maximum when $\frac{d}{d\omega}\left(\left|\frac{V_c}{V_{in}}\right|^2\right) = 0$, show that the resonance occurs at the frequency $\omega_{\text{peak}}^2 = \frac{1}{CL} - \frac{R^2}{2L^2}$.



The magnitude of V_c/V_{in} is

$$\left|\frac{V_c}{V_{in}}\right| = \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + (\omega CR)^2}}$$

which reaches its maximum under the condition $\frac{d}{d\omega}\left(\left|\frac{V_c}{V_{in}}\right|^2\right) = 0$, therefore,

$$\begin{aligned}\frac{d}{d\omega}\left|\frac{V_c}{V_{in}}\right|^2 &= \frac{d}{d\omega}\left(\frac{1}{1+\omega^4 C^2 L^2 - 2\omega^2 CL + \omega^2 C^2 R^2}\right) \\ &= -\frac{4\omega^3 CL^2 - 4\omega CL + 2\omega C^2 R^2}{\left(1+\omega^4 C^2 L^2 - 2\omega^2 CL + \omega^2 C^2 R^2\right)^2} \\ &= 0.\end{aligned}$$

Since the denominator cannot be zero, only the numerator will be zero:

$$4\omega^3 CL^2 - 4\omega CL + 2\omega C^2 R^2 = 0 \quad \rightarrow \quad \omega_{\text{peak}}^2 = \frac{1}{CL} - \frac{R^2}{2L^2}$$

For a small value of R, $\omega_{\text{peak}}^2 \approx \frac{1}{CL}$ since the term $\frac{R^2}{2L^2} \sim 0$. Substituting ω_{peak}^2 into $\left|\frac{V_c}{V_{in}}\right|^2$ yields

$$\left|\frac{V_c}{V_{in}}\right|^2_{\text{peak}} = \frac{1}{\left(1-\frac{1}{CL}\cdot LC\right)^2 + \left(\frac{1}{CL}\cdot C^2 R^2\right)} = \frac{L}{CR} \quad \rightarrow \quad \left|\frac{V_c}{V_{in}}\right|_{\text{peak}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$