Vibrations and Waves

Summary of Three Types of Signal Oscillations

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1 Free/Undamped Oscillation

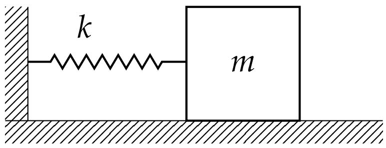


Figure 1: A spring-mass mechanical system

The equilibrium state of the system shown in Figure 1 can be mathematically described by Hooke's law:

$$F = ma = -kx \tag{1}$$

Re-arrange Equation 1, we could get

$$ma + kx = 0 \tag{2}$$

The acceleration, a, is the second-derivative of the displacement, x, with respect to the time, t: $a = \frac{d^2x}{dt^2}$. So Equation 2 can be converted to a 2^{nd}*-order ordinary differential equation*:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{3}$$

Equation 3 is commonly referred to as the *governing equation* that describes the dynamic behaviours of the mechanical system shown in Figure 1. This equation is now ready to be solved!

Solution Procedure

The coefficient of the 2^{nd}-order derivative term becomes 1 if we divide the mass m in Equation 3,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{4}$$

Applying the *trial solution* $x = A\cos(\omega t + \phi)$ to Equation 4[a]:

$$\underbrace{-A\omega^2\cos(\omega t + \phi)}_{\ddot{x}=d^2x/dt^2} + \frac{k}{m}\underbrace{A\cos(\omega t + \phi)}_{x} = 0 \tag{5}$$

Re-arrange , we could separate a common, non-zero term $\cos(\omega t + \phi)$:

$$(-\omega^2 + \frac{k}{m}) \underbrace{A \cdot \cos(\omega t + \phi)}_{\text{this term cannot be zero!}} = 0 \tag{6}$$

Equation 6 implies that only the first term $(-\omega^2 + \frac{k}{m})$ is zero (since the cosine term can *never* be zero!). Therefore, we can express ω in terms of k and m:

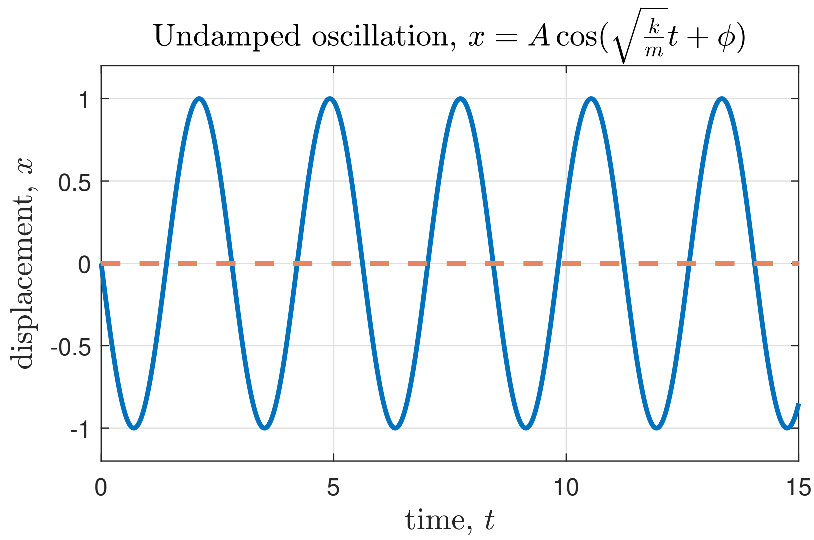
$$-\omega^2 + \frac{k}{m} = 0 \quad \rightarrow \quad \boxed{\omega = \pm\sqrt{\frac{k}{m}}} \tag{7}$$

We are only interested in the positive solution of ω! Therefore, the solution for Equation 3 is

$$\boxed{x = A\cos(\sqrt{\frac{k}{m}}t + \phi)} \tag{8}$$

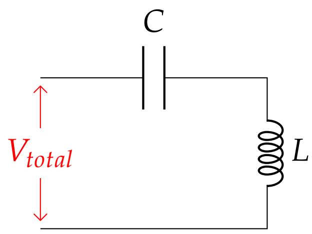
[a]Well... for now you just need to accept that this solution is correct!

Equation 8 is the general solution for a **undamped system**. Let us visualise this by plotting the displacement as a function of time (x-t):



As you can see, there is no decay of the displacement as time goes by, *i.e.*, the amplitude of the displacement is constant due to the absence of the damping effects. The mass in the spring-mass system will move back and forth with perfect conservation of energy!

Electrical analogy The electrical equivalent circuit that can generate the free oscillation is a capacitance (C) -inductance (L) circuit.



The voltage across

- the inductor, L: $V_L = L\frac{dI}{dt}$
- the capacitor, C: $V_C = \frac{1}{C}\int_0^t I(\tau)d\tau$

By Kirchhoff's voltage law:

$$V_L + V_C = V_{total} \quad \rightarrow \quad L\frac{dI}{dt} + \frac{1}{C}\int_0^t I(\tau)d\tau = V_{total}$$

Differentiate:

$$L\frac{d^2I}{dt^2} + \frac{1}{C}I = 0$$

which is the governing equation for the above L-C system.

2 Damped Oscillation

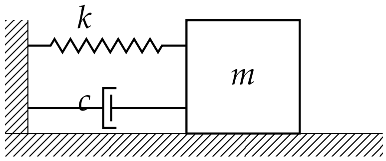


Figure 2: A spring-mass-damper mechanical system

Figure 2 shows a spring-mass-damper (*k*-*m*-*c*) system. The equilibrium state of the system can be mathematically described by:

$$F = ma = -cv - kx \tag{9}$$

where c is known as the *damping coefficient*, v is the moving speed of the mass, and $c \cdot v$ is defined as the force exerted by the mechanical damper. Re-arrange Equation 9, we could get

$$ma + cv + kx = 0 \tag{10}$$

The velocity, v and the acceleration, a are defined as the 1st and 2nd derivative of the displacement, x, with respect to the time, t: $v = \frac{dx}{dt}$, $a = \frac{d^2x}{dt^2}$. Therefore, Equation 10 can be converted to a *2nd-order ordinary differential equation* (again!)

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{11}$$

Solution Procedure

The coefficient of the 2nd-order derivative term becomes 1 if we divide the mass m in Equation 11,

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{12}$$

Let us first define two parameters: *natural frequency* and *damping factor*:

- Natural frequency, $\omega_n = \sqrt{\frac{k}{m}}$
- Damping factor, $\gamma = \frac{c}{2\sqrt{km}}$

If we apply the natural frequency and damping factor defined above to Equation 12, we will obtain a more *generic* expression of the governing equation:

$$\frac{d^2x}{dt^2} + 2\gamma\omega_n\frac{dx}{dt} + \omega_n^2 x = 0 \tag{13}$$

To solve Equation 13, we shall apply the trial solution $x = Ae^{\mu t}$ to Equation 13[a].

$$\underbrace{\mu^2 Ae^{\mu t}}_{\ddot{x}} + 2\gamma\omega_n \underbrace{\mu Ae^{\mu t}}_{\dot{x}} + \omega_n^2 \underbrace{Ae^{\mu t}}_{x} = 0 \tag{14}$$

Re-arrange , we could separate a common, non-zero term $Ae^{\mu t}$:

$$(\mu^2 + 2\gamma\omega_n\mu + \omega_n^2)\underbrace{A \cdot e^{\mu t}}_{\text{non-zero!}} = 0 \tag{15}$$

Equation 15 implies that only the first term $(\mu^2 + 2\gamma\omega_n\mu + \omega_n^2)$ is zero (since the exponential term can *never* be zero!). Therefore, we can solve the quadratic equation of μ to solve μ in terms of γ and ω_n:

$$\mu^2 + 2\gamma\omega_n\mu + \omega_n^2 = 0 \quad \rightarrow \quad \boxed{\mu = -\gamma\omega_n \pm \omega_n\sqrt{\gamma^2 - 1}} \tag{16}$$

What is the general solution to displacement? **we need to consider three conditions of** $\gamma^2 - 1$ (the term under the square root), **as they correspond to 3 different types of damping effects.**

[a] Let us assume this trial solution is correct for now!

2.1 Condition 1: $\gamma^2 - 1 < 0$ - Light Damping

If $\gamma^2 - 1 < 0$, the general solution of a 2nd-order ODE should hold the format

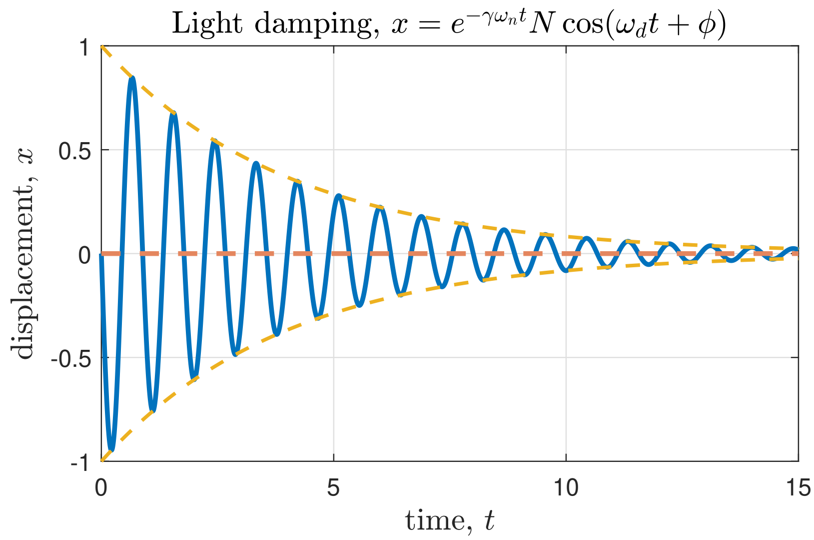
$$x = e^{\mu t}(A_1 \cos(\omega_n x) + iA_2 \sin(\omega_n x))$$

i.e., the solution *might be* a complex number.

To determine this, for convenience, we first define $\omega_d = \omega_n\sqrt{1 - \gamma^2}$,

$$\begin{aligned}
x &= A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} \\
&= A_1 e^{(-\gamma\omega_n + j\omega_d)t} + A_2 e^{(-\gamma\omega_n - j\omega_d)t} \\
&= e^{-\gamma\omega_n t}\Big(A_1\big(\cos(\omega_d t) + j\sin(\omega_d t)\big) + A_2\big(\cos(\omega_d t) - j\sin(\omega_d t)\big)\Big) \\
&= e^{-\gamma\omega_n t}\Big[\underbrace{(A_1 + A_2)}_{C = N\cos\phi}\cos(\omega_d t) + \underbrace{j(A_1 - A_2)}_{-D = -N\sin\phi}\sin(\omega_d t)\Big] \\
&= e^{-\gamma\omega_n t}\big(N\cos\phi\cos\omega_d t - N\sin\phi\sin\omega_d t\big) \\
&= \boxed{e^{-\gamma\omega_n t}N\cos(\omega_d t + \phi)}
\end{aligned} \tag{17}$$

To plot x against t:



Two observations we can make here:

1. the occurrence of oscillations, this is described by the cosine term in Equation 17; and
2. the amplitude of oscillation decays with time (damped) - this is due to the exponential term in Equation 17. The yellow envelops shown in Equation 2.1 are exactly the plot of $e^{-\omega_n \gamma t}$ and $-e^{-\omega_n \gamma t}$.

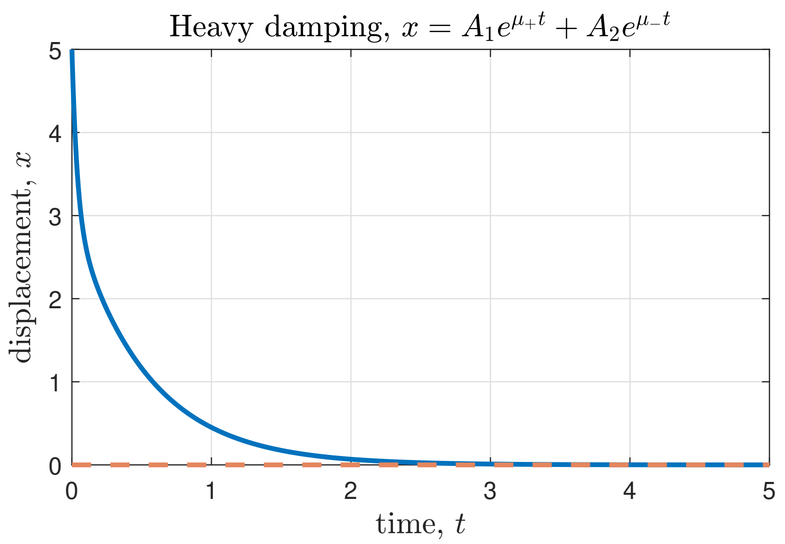
This type of damping oscillation is commonly known as the **light damping**.

2.2 Condition 2: $\gamma^2 - 1 > 0$ - Heavy Damping

If $\gamma^2 - 1 > 0$, there are two distinct roots of μ, therefore, the general solution becomes

$$\boxed{x = A_1 e^{\mu_+ t} + A_2 e^{\mu_- t} = A_1 e^{-(\gamma\omega_n + \omega_n \sqrt{\gamma^2 - 1})t} + A_2 e^{-(\gamma\omega_n - \omega_n \sqrt{\gamma^2 - 1})t}} \tag{18}$$

To plot x against t:



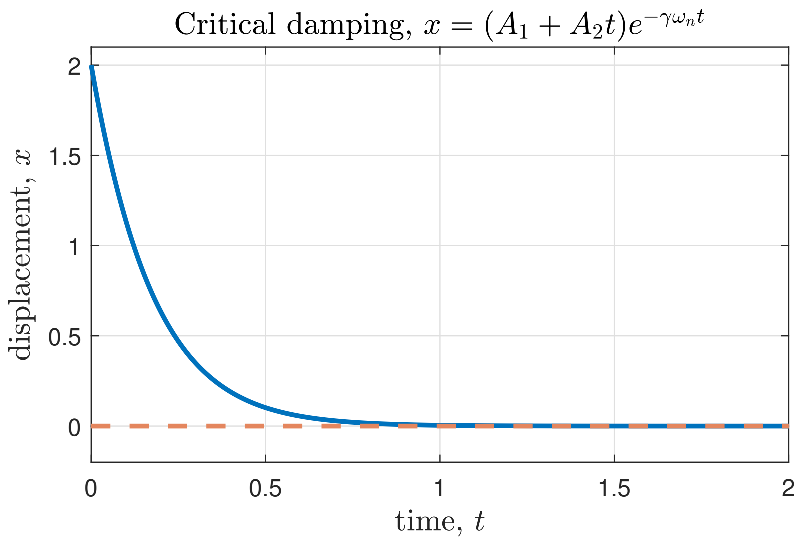
The mass attains its equilibrium gradually **without** any oscillation. This is known as the **heavy damping**.

2.3 Condition 3: $\gamma^2 - 1 = 0$ - Critical Damping

If $\gamma^2 - 1 = 0$, the general solution of a 2[nd]-order ODE should hold the format

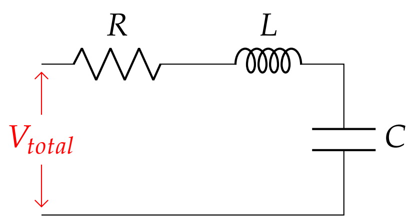
$$\boxed{x = (A_1 + A_2 t)e^{\mu t}} \tag{19}$$

where in this situation, $\mu = -\omega_n \gamma$. To plot x against t:



The mass returns to the equilibrium position as quickly as possible (*i.e.*, quickly within 1 oscillation). This is known as the **critical damping**. It is the threshold between heavy damping and light damping.

Electrical analogy Three types of damped oscillations can be generated with the following L-C-R circuit:



The voltage across

- the resistor, R: $V_r = RI$
- the inductor, L: $V_L = L\dfrac{dI}{dt}$
- the capacitor, C: $V_C = \dfrac{1}{C}\displaystyle\int_0^t I(\tau)d\tau$

By Kirchhoff's voltage law:

$$V_R + V_L + V_C = V_{total} \quad \rightarrow \quad RI + L\frac{dI}{dt} + \frac{1}{C}\int_0^t I(\tau)d\tau = V_{total}$$

Differentiate:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$

which is a second-order, homogeneous differential equation. To solve this ODE, the corresponding auxiliary equation is

$$Lm^2 + Rm + \frac{1}{C} = 0$$

which yields two solutions

$$m_{1,2} = \frac{-R}{2L} \pm \frac{\sqrt{R^2 - 4L/C}}{2L}$$

we need to discuss the value of $\sqrt{R^2 - 4L/C}$ to determine the type of damping:

- $R^2 > 4L/C$: heavy damping
- $R^2 < 4L/C$: under damping
- $R^2 = 4L/C$: critical damping

3 Forced Oscillation

So far, we have considered both undamped oscillation and damped oscillation - no further external force was applied once the system was released from the initial position. How will system dynamic behaves if we apply an external force to the system periodically?

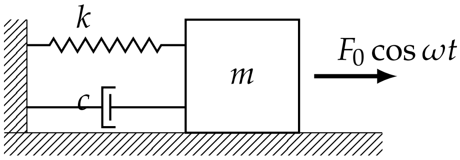


Figure 3: A spring-mass-damper mechanical system subjected to a periodic force

Figure 3 shows a spring-mass-damper (*k*-*m*-*c*) system subjected to a periodic external force $F(t) = F_0 \cos \omega t$. The equilibrium state of the system can be mathematically described by:

$$F_{total} = ma = -cv - kx + F_0 \cos \omega t \tag{20}$$

Re-arrange Equation 20, we could get

$$ma + cv + kx = F_0 \cos \omega t \tag{21}$$

Similarly, the velocity, v and the acceleration, a are defined as the 1st and 2nd derivative of the displacement, x, with respect to the time, t: $v = \frac{dx}{dt}$, $a = \frac{d^2x}{dt^2}$. Therefore, Equation 10 can be converted to an *inhomogeneous* 2nd-order ordinary differential equation.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \cos \omega t \tag{22}$$

and it is equivalent to

$$\frac{d^2x}{dt^2} + 2\gamma\omega_n\frac{dx}{dt} + \omega_n^2 x = \frac{F_0}{m} \cos \omega t \tag{23}$$

where ω_n is natural frequency, γ is damping factor[1].

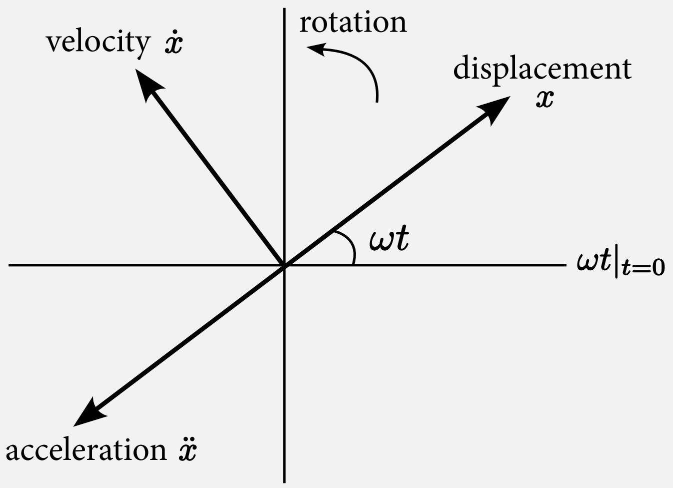
Solution Procedure and Discussions

To solve this ODE, we shall apply the trial solution $x = x_0 \cos(\omega t - \phi)$. Therefore, we have:

- the velocity, $v = \frac{dx}{dt} = -x_0\omega \sin(\omega t - \phi) = x_0\omega \cos(\omega t - \phi + \frac{\pi}{2})$; and
- the acceleration, $a = \frac{d^2x}{dt^2} = -x_0\omega^2 \cos(\omega t - \phi) = x_0\omega^2 \cos(\omega t - \phi + \pi)$;

The relative location of x, v, and a can be roughly plotted as

[1] Why? See the previous chapter.



i.e., there exists a $\pi/2$ and π phase difference between the velocity and displacement, and acceleration and displacement, respectively.

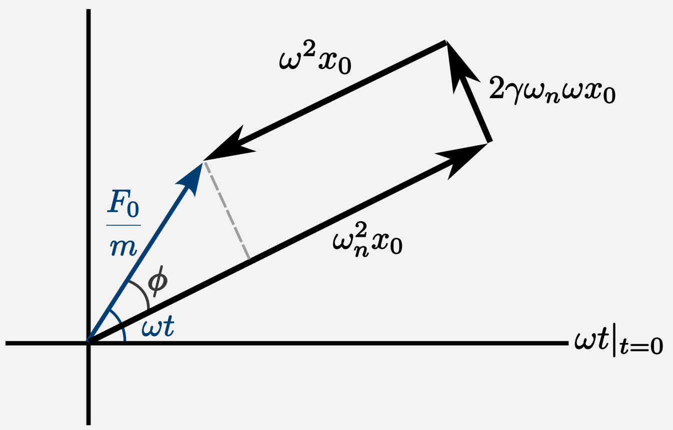
Expression the trial solution in Euler's form: $x = x_0 \cos(\omega t - \phi) = x_0 e^{j(\omega t - \phi)}$ and substitute into the governing equation:

$$\underbrace{-\omega^2 x_0 e^{j\omega t} e^{-j\phi}}_{\ddot{x}} + 2\gamma\omega_n \underbrace{j\omega x_0 e^{j\omega t} e^{-j\phi}}_{\dot{x}} + \omega_n^2 \underbrace{x_0 e^{j\omega t} e^{-j\phi}}_{x} = \frac{F_0}{m} e^{j\omega t} \tag{24}$$

Re-arrange,

$$\cancel{e^{j\omega t}} e^{-j\phi} (-\omega^2 x_0 + j2\gamma\omega_n \omega x_0 + \omega_n^2 x_0) = \frac{F_0}{m} \cancel{e^{j\omega t}} \tag{25}$$

What does Equation 25 tell us? Well, the term $e^{-\phi j}$ implies an anticlockwise rotation of an angle ϕ. Therefore, Equation 25 can be graphically represented as



The length of F_0/m (blue) can be parsed into 3 individual trajectories - $\omega_n^2 x_0$ is the trajectory of the displacement, $2\gamma\omega_n\omega x_0$ is the trajectory of the velocity, and $\omega^2 x_0$ is the trajectory of the acceleration[a].

Therefore, we can represent the length (*magnitude*) of x_0 using Pythagoras' theorem,

$$|x_0| = \frac{F_0/m}{\sqrt{(\omega_n^2 \omega^2)^2 + (2\gamma\omega_n\omega)^2}} \tag{26}$$

and similarly, the *phase* of x_0:

$$\tan\phi = \frac{2\gamma\omega_n\omega}{\omega_n^2 - \omega^2} \quad \rightarrow \quad \phi = \tan^{-1}\left(\frac{2\gamma\omega_n\omega}{\omega_n^2 - \omega^2}\right) \tag{27}$$

Let us think about how the relations between ω, ω_n, $2\gamma\omega_n\omega$ could affect the magnitude response:

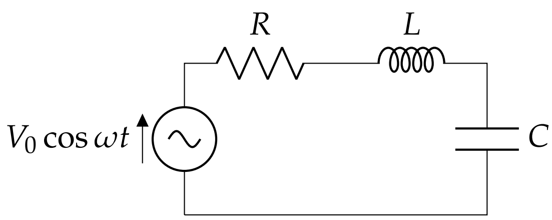
1. When $\omega \to 0$,
$$x_0 = \frac{F_0}{m\omega_n^2} = \frac{F_0}{k}$$
which implies that x_0 is stiffness controlled.

2. If $\omega \to \infty$,
$$\omega^2 = \frac{F_0}{mx_0} \quad \rightarrow \quad x_0 = \frac{F_0}{\omega^2 m}$$
which implies that x_0 is mass controlled.

3. If $\omega_n = \omega$, the gradient of the magnitude response $|x_0|$ with respect to ω is zero, the peak magnitude of x_0 occurs - also known as the **resonance**. The **resonance frequency** is represented as:
$$\omega_p = \sqrt{1 - 2\gamma^2}$$

[a]Note their directions, and correlates to the figure above!

Electrical analogy The forced oscillation can be generated with the following L-C-R circuit with an additional periodic voltage source term:



The governing question of the system shown above is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = V_0\cos\omega t$$