Pre-sessional learning for iBSc students
Section 7: Definite integrals in multiple dimensions

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Question 1: calculate $\int_1^2 \mathrm{d}y \int_1^3 \mathrm{d}x \; x/y$.

1 - Integrate from left to right: the first integral to find is $\int_1^3 \frac{x}{y}\,\mathrm{d}x$, hold y as a constant,

$$\int_1^3 \frac{x}{y}\,\mathrm{d}x = \left[\frac{x^2}{2y}\right]_1^3 = \frac{4}{y}.$$

2 - Integrate the result w.r.t. y,

$$\int_1^2 \frac{4}{y}\mathrm{d}y = [4\ln(y)]_1^2 = 4\ln 2.$$

Question 2: calculate $\int_{-\pi}^{\pi} \mathrm{d}u \int_{-\pi/2}^{\pi/2} \mathrm{d}v \int_{-1}^{1} \mathrm{d}w \; w\cos(u-v)$.

1 - The first integral to find is $\int_{-1}^{1} w\cos(u-v)\mathrm{d}w$, with u and v being held constant:

$$\int_{-1}^{1} w\cos(u-v)\mathrm{d}w = \left[\frac{1}{2}w^2\cos(u-v)\right]_{-1}^{1} = 0.$$

(This is because the symmetry of $\frac{1}{2}w^2\cos(u-v)$ and limits.)

2 - Due to the inner integral being 0, we are confident that the rest should be 0. There is no need to evaluate the remaining integrals further.

Question 3: calculate $\int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\ y \cdot \exp(-zy)$.

1 - The first integral to find is $\int_0^1 y \cdot \exp(-zy)\ \mathrm{d}z$, with y being held constant:

$$\int_0^1 y \cdot \exp(-zy)\ \mathrm{d}z = \left[y \cdot -\frac{1}{y} \cdot \exp(-zy)\right]_0^1 = -\exp(-y) + 1.$$

2 - Integrate the result w.r.t. y,

$$\int_0^1 (\exp(-y) + 1)\mathrm{d}y = \left[\exp(-y) + y\right]_0^1 = \exp(-1) + 1 - 1 = \exp(-1).$$

Question 4: calculate $\int_0^1 \mathrm{d}y \int_y^1 \mathrm{d}x\ (x - y)^2$.

1 - The first integral to find is $\int_y^1 (x - y)^2\ \mathrm{d}x$, with y being held constant:

$$\int_y^1 (x - y)^2\ \mathrm{d}x = \left[\frac{1}{3}(x - y)^3\right]_y^1 = \frac{1}{3}(1 - y)^3.$$

2 - Integrate the result w.r.t. y,

$$\int_0^1 \frac{1}{3}(1 - y)^3 \mathrm{d}y = \left[-\frac{1}{12}(1 - y)^4\right]_0^1 = \frac{1}{12}.$$

Question 5: calculate the volume between the surface $f(x, y) = x^2y^2$ and the x-y plane, restricted to the range $0 \leq x < 1$ and $0 \leq y < 1 - x$.

1 - As described, we can formulate the volume integral as

$$\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\ x^2y^2.$$

Why let the y be the inner integral, but not the opposite? You may note that the range of y is bounded by x (*i.e.*, $0 \leq y \leq 1 - x$). Hence, evaluating the integral w.r.t. y first ensures the domain of y is fully captured by the subsequent integral w.r.t. x.

2 - The first integral to find is $\int_0^{1-x} x^2y^2\ \mathrm{d}y$, with x being held constant:

$$\int_0^{1-x} x^2y^2\ \mathrm{d}y = \left[\frac{1}{3}x^2y^3\right]_0^{1-x} = \frac{1}{3}x^2(1 - x)^3.$$

3 - Integrate the result w.r.t. x,

$$\int_0^1 \frac{1}{3}x^2(1 - x)^3 \mathrm{d}x.$$

By the hint, an expansion can be used to simplify the question,

$$x^2(1-x)^3 = x^2(1-3x+3x^2-x^3) = x^2 - 3x^3 + 3x^4 - x^5.$$

Hence, the original integral is

$$\frac{1}{3}\int_0^1 x^2 - 3x^3 + 3x^4 - x^5 \mathrm{d}x = \frac{1}{3}\cdot\left[\frac{1}{3}x^3 - \frac{3}{4}x^4 + \frac{3}{5}x^5 - \frac{1}{6}x^6\right]_0^1 = \frac{1}{180}.$$

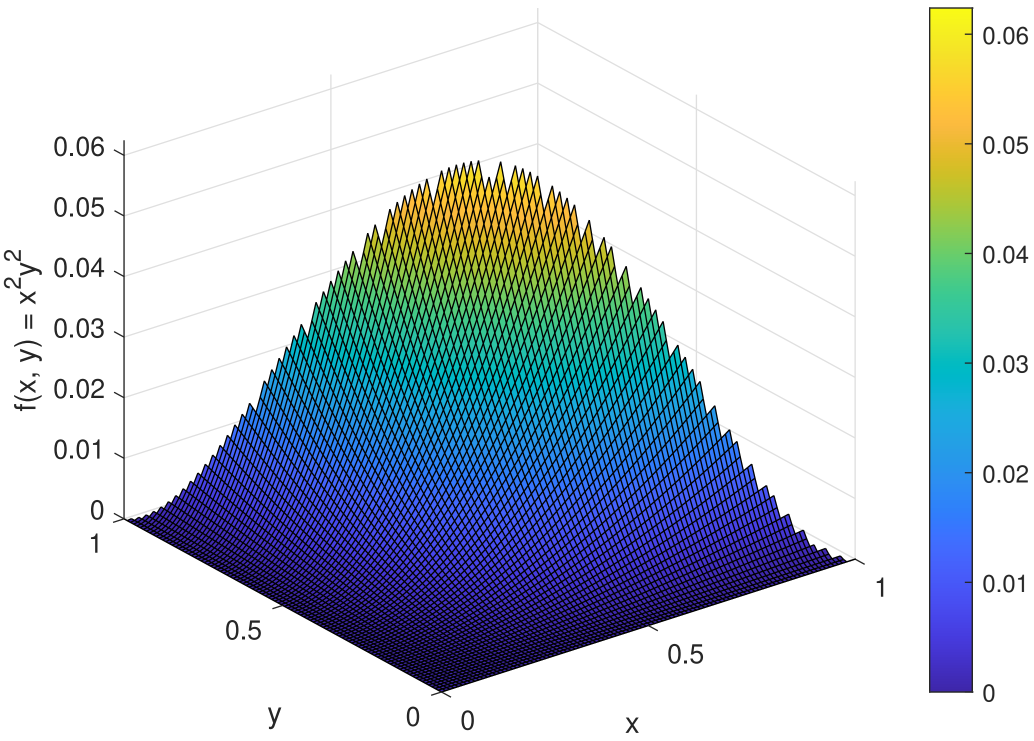

Figure 1: Surface plot of $f(x, y) = x^2 y^2$ as per described in Question 5. Note that the jagged edges are attributed to the bounded range of $y : \{0 \leq y < 1 - x\}$ that masked out the elements outside this range.