Pre-sessional learning for iBSc students
Section 7: Higher order partial derivatives

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Rules for computing partial derivatives:

- same rules as the differentiation of single variable functions.
- while differentiating a function of several variables with respect to one independent variable, we keep all other independent variables as constants (or as coefficients).

Question 1: Find $\frac{\partial^2 f}{\partial x^2}$ of the function $f(x, y) = 3x^2 y + x\sin(y)$.

1 - find the first derivative of f w.r.t. x, while y is held constant: $\frac{\partial f}{\partial x} = 6xy + \sin(y)$.

2 - find the second derivative of f w.r.t. x: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6y$.

Question 2: Find $\frac{\partial^2 g}{\partial u^2}$ of the function $g(u, v) = \exp\left(\frac{1}{3}(u-v)^2\right)$.

1 - find the first derivative of g w.r.t. u, while v is held constant:

$$\frac{\partial g}{\partial u} = \frac{1}{3}\cdot 2(u-v)\cdot \exp\left(\frac{1}{3}(u-v)^2\right).$$

2 - find the second derivative of g w.r.t. u (differentiation by part):

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial u}\right)\\ &= \frac{2}{3}\exp\left(\frac{1}{3}(u-v)^2\right) + \frac{2}{3}(u-v)\cdot\exp\left(\frac{1}{3}(u-v)^2\right)\cdot\frac{2}{3}(u-v)\\ &= \left(\frac{2}{3}+\frac{4}{9}(u-v)^2\right)\exp\left(\frac{1}{3}(u-v)^2\right).\end{aligned}$$

Question 3: Find $\dfrac{\partial^2 f}{\partial v \partial w}$ of the function $f(u, v, w) = w \ln(u \cdot v/w) + v \cos(w)$.

1 - for simplicity in the subsequent calculations, we can decompose the fraction in the natural logarithm

$$\ln\left(\frac{u \cdot v}{w}\right) \equiv \ln(u \cdot v) - \ln(w).$$

Hence,

$$f(u, v, w) = w \cdot [\ln(u \cdot v) - \ln(w)] + v \cos(w).$$

2 - find the first derivative of f w.r.t. w, while u and v are held constant:

$$\begin{aligned} \frac{\partial f}{\partial w} &= [\ln(u \cdot v) - \ln(w)] + w \cdot \left(-\frac{1}{w}\right) - v \sin(w) \\ &= [\ln(u \cdot v) - \ln(w)] - 1 - v \sin(w) \end{aligned}$$

3 - find the derivative of $\dfrac{\partial f}{\partial w}$ w.r.t. v, while u and w are held constant:

$$\frac{\partial^2 f}{\partial v \partial w} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial w}\right) = \frac{1}{v} - \sin(w)$$

Question 4: Find $\dfrac{\partial^3 w}{\partial u^3}$ of the function $w(u, v) = (2u - v)^6 - u^2 \tan(v)$.

1 - find the first derivative of w w.r.t. u, while v is held constant:

$$\frac{\partial w}{\partial u} = 6(2u - v)^5 \cdot 2 - 2u \tan(v) = 12(2u - v)^5 - 2u \tan(v).$$

2 - find the second derivative of w w.r.t. u, while v is held constant:

$$\frac{\partial^2 w}{\partial u^2} = 60(2u - v)^4 \cdot 2 - 2\tan(v) = 120(2u - v)^4 - 2\tan(v).$$

3 - find the third derivative of w w.r.t. u, while v is held constant:

$$\frac{\partial^3 w}{\partial u^3} = 480(2u - v)^3 \cdot 2 = 960(2u - v)^3.$$

Question 5: Find $\dfrac{\partial^3 q}{\partial r \partial t \partial s}$ of the function $q(r, s, t) = 6tr^2 \cos(rs)$.

1 - find the derivative of q w.r.t. s, while r and t are held constant:

$$\frac{\partial q}{\partial s} = -6tr^3 \sin(rs).$$

2 - find the derivative of $\dfrac{\partial q}{\partial s}$ w.r.t. t, while r and s are held constant:

$$\frac{\partial^2 q}{\partial t \partial s} = \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial s}\right) = -6r^3 \sin(rs).$$

3 - find the derivative of $\frac{\partial^2 q}{\partial t \partial s}$ w.r.t. r, while t and s are held constant:

$$\frac{\partial^3 q}{\partial r \partial t \partial s} = \frac{\partial}{\partial r}\left(\frac{\partial^2 q}{\partial t \partial s}\right) = -18r^2 \sin(rs) - 6sr^3 \cos(rs).$$