Pre-sessional learning for iBSc students
Section 7: Stationary points in two dimensions

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The discriminant, D, used to check the type o f stationary point of a function $f(x, y)$ is defined as

$$D = \frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2,$$

where a stationary point can be classified as $\begin{cases} \text{local maxima or minima}, & \text{if } D > 0 \\ \text{saddle}, & \text{if } D < 0 \\ \text{inconclusive}, & \text{if } D = 0 \end{cases}$.

Question 1: Let $g(u, v) = (u-1)^2 + v\exp(-v)$. Identify the values of u and v corresponding to the stationary points of the function.

- Calculate $\frac{\partial g(u,v)}{\partial u}$, find u and/or v that satisfies $\frac{\partial g(u,v)}{\partial u} = 0$:

$$\frac{\partial g(u,v)}{\partial u} = 2(u-1) = 0 \quad \Rightarrow \quad u = 1.$$

- Calculate $\frac{\partial g(u,v)}{\partial v}$, find u and/or v that satisfies $\frac{\partial g(u,v)}{\partial v} = 0$:

$$\frac{\partial g(u,v)}{\partial v} = (v-1)e^{-v} = 0 \quad \Rightarrow \quad v = 1.$$

- Hence, the stationary point is $(1, 1)$.

Question 2: Let $f(x, y) = 3\cos(x) + \sin(2y - \pi)$. Identify the values of x and y corresponding to the stationary points of the function.

- Calculate $\frac{\partial f(x,y)}{\partial x}$, find x and/or y that satisfies $\frac{\partial f(x,y)}{\partial x} = 0$:

$$\frac{\partial f(x,y)}{\partial x} = -3\sin(x) = 0 \quad \Rightarrow \quad x = n\pi, \text{ where } n \text{ is an integer.}$$

- Calculate $\frac{\partial f(x,y)}{\partial y}$, find x and/or y that satisfies $\frac{\partial f(x,y)}{\partial y} = 0$:

$$\frac{\partial f(x,y)}{\partial y} = 2\cos(2y-\pi) = 0 \quad \Rightarrow \quad 2y-\pi = \underbrace{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...}_{(2n-1)\frac{\pi}{2},\ n \text{ is an integer}} \quad \Rightarrow \quad y = (2n+1)\frac{\pi}{4}.$$

- Hence, the stationary point is $\left(n\pi, \frac{(2n+1)\pi}{4}\right)$, where n is an integer.

Question 3: Let $f(x,y) = xye^{-x-2y}$. Identify the values of x, and y corresponding to the stationary points of the function.

- Calculate $\frac{\partial f(x,y)}{\partial x}$, find x and/or y that satisfy $\frac{\partial f(x,y)}{\partial x} = 0$:

$$\frac{\partial f(x,y)}{\partial x} = y(1-x)e^{-x-2y} = 0 \quad \Rightarrow \quad y = 0 \text{ or } x = 1.$$

- Calculate $\frac{\partial f(x,y)}{\partial y}$, find x and/or y that satisfies $\frac{\partial f(x,y)}{\partial y} = 0$:

$$\frac{\partial f(x,y)}{\partial y} = x(1-2y)e^{-x-2y} = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = 1/2.$$

- Hence, the stationary points is $(0,0)$ and $(1,1/2)$.

[You may wonder, why not $(1,0)$ or $(0,1/2)$? Plug these pairs into the partial derivatives we have obtained above, and you will notice they would not result in a zero derivative.]

Question 4: $g(u,v) = \cos(u/v) + (v-1)^2$ has a stationary point at $u = \pi$, $v = 1$. Evaluate the quantity D at $u = \pi$, $v = 1$, and use it to identify the nature of the stationary point.

- Calculate $\frac{\partial^2 g(u,v)}{\partial u^2}$

$$\frac{\partial g(u,v)}{\partial u} = -\sin(u/v)\cdot(1/v)$$
$$\frac{\partial^2 g(u,v)}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial g(u,v)}{\partial u}\right) = -\frac{1}{v^2}\cos(u/v).$$

Hence, $\left.\frac{\partial^2 g(u,v)}{\partial u^2}\right|_{u=\pi,v=1} = 1.$

- Calculate $\frac{\partial^2 g(u,v)}{\partial v^2}$

$$\frac{\partial g(u,v)}{\partial v} = (u/v^2)\sin(u/v) + 2(v-1)$$
$$\frac{\partial^2 g(u,v)}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial g(u,v)}{\partial v}\right) = -\frac{2u}{v^3}\sin(u/v) - \frac{u^2}{v^4}\cos(u/v) + 2.$$

Hence, $\frac{\partial^2 g(u,v)}{\partial v^2}\Big|_{u=\pi,v=1} = \pi^2 + 2.$

- Calculate $\frac{\partial^2 g(u,v)}{\partial u \partial v}$

$$\frac{\partial^2 g(u,v)}{\partial u \partial v} = \frac{1}{v^2}\sin(u/v) + \frac{u}{v^3}\cos(u/v).$$

Hence, $\frac{\partial^2 g(u,v)}{\partial u \partial v}\Big|_{u=\pi,v=1} = -\pi.$

- Evaluate discriminant D:

$$D = (1)\cdot(\pi^2+2) - (-\pi)^2 = 2 > 0 \quad \Rightarrow \quad \text{this is a local minima.}$$

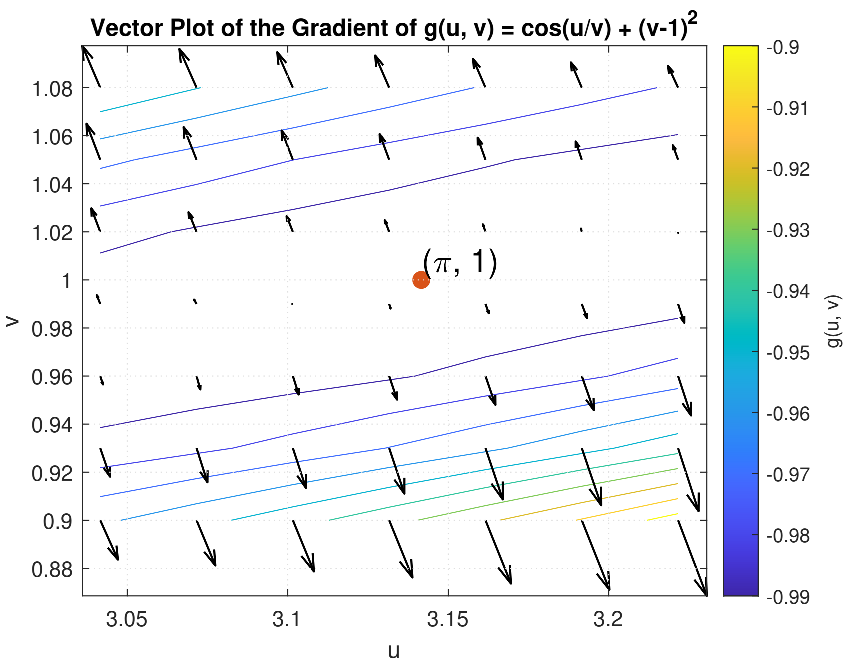

Figure 1: Vector plot of the gradient of $g(x, y)$ as defined in Question 4.

Question 5: $f(x,y) = e^{(x+3)^2}\, e^{-(2y-1)^2}$ **has a stationary point at** $x = -3, y = 1/2$**. Evaluate the quantity** D **at** $x = -3$, $y = 1/2$**, and use it to identify the nature of the stationary point.**

- Calculate $\dfrac{\partial^2 f}{\partial x^2}$:

$$\frac{\partial f}{\partial x} = 2(x+3)\, e^{(x+3)^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = (2 + 4(x+3)^2)\, e^{(x+3)^2}$$

Hence, $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{x=-3,y=0.5} = 2.$

- Calculate $\dfrac{\partial^2 f}{\partial y^2}$:

$$\frac{\partial f}{\partial y} = -4(2y-1)\, e^{-(2y-1)^2} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = -(8 + 16(2y-1)^2)\, e^{-(2y-1)^2},$$

Hence, $\left.\dfrac{\partial^2 f}{\partial y^2}\right|_{x=-3,y=0.5} = -8.$

- Mixed second partial derivative $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ since $\dfrac{\partial f}{\partial y}$ is invariant of x.

- Evaluate discriminant D:

$$D = (2) \cdot (-8) - (0)^2 = -16 \; < 0 \quad \Rightarrow \quad \text{this is a saddle point.}$$

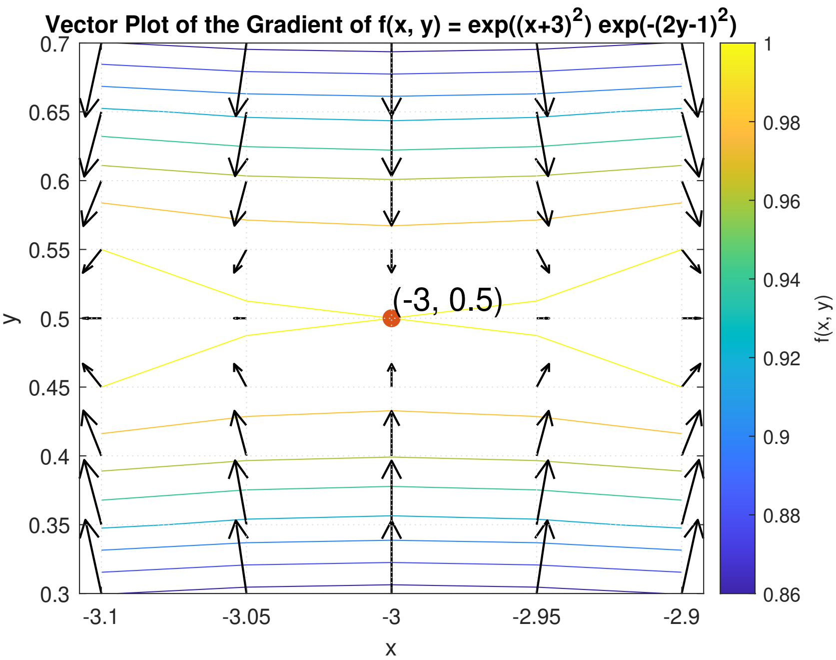

Figure 2: Vector plot of the gradient of $f(x,y)$ as defined in Question 5.