Pre-sessional learning for iBSc students
Section 7: Total derivatives and gradients

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The total derivative, df, of a multi-variable function, $f(x, y)$, is given by

$$df = \frac{\partial f(x, y)}{\partial x}\, dx + \frac{\partial f(x, y)}{\partial y}\, dy,$$

where dx, dy represent the *small* changes in x and y, respectively. Hence, the total derivative combines the effects of changes of f in both variables in the neighbourhood of a point, *i.e.*, from (x, y) to $(x + dx, y + dy)$.

Question 1: Let $f(x, y) = e^{3x} \ln(2y)$. By calculating the total derivative, find the approximate change in f as x, y undergoes a small change from $(1, 1)$ to $(1 + dx, 1 + dy)$.

- Calculate $\frac{\partial f(x, y)}{\partial x}$:

$$\frac{\partial f(x, y)}{\partial x} = 3e^{3x} \ln(2y).$$

- Calculate $\frac{\partial f(x, y)}{\partial y}$:

$$\frac{\partial f(x, y)}{\partial x} = e^{3x} \frac{1}{y}.$$

- Therefore, the total derivative is given by

$$df = 3e^{3x} \ln(2y)\, dx + e^{3x} \frac{1}{y}\, dy.$$

By substituting $(1, 1)$ into df gives us the change in f from $(1, 1)$ to $(1 + dx, 1 + dy)$:

$$df\Big|_{x=1,y=1} = (3e^3 \ln(2))\, dx + (e^3)\, dy.$$

Question 2: Let $f(u,v,w) = u^3/w - uv\sin(v^2)$. By calculating the total derivative, find the approximate change in f as u, v, w undergoes a small change from $(1,2,2)$ to $(1+du, 2+dv, 2+dw)$.

- Calculate $\dfrac{\partial f(u,v,w)}{\partial u}$:

$$\frac{\partial f(u,v,w)}{\partial u} = 3u^2/w - v\sin(v^2).$$

- Calculate $\dfrac{\partial f(u,v,w)}{\partial v}$:

$$\frac{\partial f(u,v,w)}{\partial v} = -u\sin(v^2) - 2uv^2\cos(v^2).$$

- Calculate $\dfrac{\partial f(u,v,w)}{\partial w}$:

$$\frac{\partial f(u,v,w)}{\partial w} = -u^3/w^2.$$

- Therefore, the total derivative is given by

$$df = (3u^2/w - v\sin(v^2))\,du - (u\sin(v^2) + 2uv^2\cos(v^2))\,dv - (u^3/w^2)\,dw.$$

By substituting $(1,2,2)$ into df gives us the change in f from $(1,2,2)$ to $(1+du, 2+dv, 2+dw)$:

$$df\Big|_{u=1,v=2,w=2} = \left(\frac{3}{2} - 2\sin(4)\right)\,du - (\sin(4) + 8\cos(4))\,dv - \frac{1}{4}\,dw.$$

The gradient of a scalar c is defined as

$$\nabla c = \frac{\partial c}{\partial x}\hat{\mathbf{x}} + \frac{\partial c}{\partial y}\hat{\mathbf{y}} + \frac{\partial c}{\partial x}\hat{\mathbf{z}}\,,$$

where ∇ ('nabla') is the gradient operator, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are the unit vectors in three orthonormal directions in the Cartesian coordinate system. Note that: c is a scalar, but ∇c is a vector.

Question 3: Calculate the two-dimensional vector $\nabla f(x,y)$, where $f(x,y) = 4x\ln(1+x^2/y)$ and x, y are Cartesian coordinates.

- Calculate $\dfrac{\partial f(x,y)}{\partial x}$:

$$\frac{\partial f(x,y)}{\partial x} = 4\ln(1+x^2/y) + 4x \cdot \frac{2x}{1+x^2} = 4\ln(1+x^2/y) + \frac{8x^2}{1+x^2}$$

- Calculate $\dfrac{\partial f(x,y)}{\partial y}$: to ease the calculation, we can rearrange the expression of $f(x,y)$:

$$f(x,y) = 4x\ln(1+x^2/y) = 4x\ln\left(\frac{y+x^2}{y}\right) = 4x[\ln(y+x^2) - \ln y]^2.$$

Hence,

$$\frac{\partial f(x,y)}{\partial y} = 4x\left(\frac{1}{y+x^2} - \frac{1}{y}\right) = 4x \cdot \frac{-x^2}{y(y+x^2)} = \frac{-4x^3}{y(y+x^2)}.$$

Therefore,

$$\nabla f(x,y) = \left[4\ln(1+x^2/y) + \frac{8x^2}{1+x^2}\right]\hat{\mathbf{x}} + \left[\frac{-4x^3}{y(y+x^2)}\right]\hat{\mathbf{y}}.$$

(Alternatively, you can arrange this solution using a column vector, as shown by the solution on the sheet.)

Question 4: Calculate the three-dimensional vector $\nabla f(x,y,z)$, where $f(x,y) = \dfrac{\sin(xy+2z)}{x^2-y^2}$ and x, y, z are Cartesian coordinates.

- Calculate $\dfrac{\partial f(x,y,z)}{\partial x}$:

$$\begin{aligned}\frac{\partial f(x,y,z)}{\partial x} &= \frac{1}{x^2-y^2}\cdot y\cdot\cos(xy+2z) \;+\; \sin(xy+2z)\cdot -2x\cdot(x^2-y^2)^{-2}\\ &= \frac{y(x^2-y^2)\cos(xy+2z) - 2x\sin(xy+2z)}{(x^2-y^2)^2}.\end{aligned}$$

- Calculate $\dfrac{\partial f(x,y,z)}{\partial y}$:

$$\begin{aligned}\frac{\partial f(x,y,z)}{\partial y} &= \frac{1}{x^2-y^2}\cdot x\cdot\cos(xy+2z) \;+\; \sin(xy+2z)\cdot 2y\cdot(x^2-y^2)^{-2}\\ &= \frac{x(x^2-y^2)\cos(xy+2z) + 2y\sin(xy+2z)}{(x^2-y^2)^2}.\end{aligned}$$

- Calculate $\dfrac{\partial f(x,y,z)}{\partial z}$:

$$\frac{\partial f(x,y,z)}{\partial z} = \frac{1}{x^2-y^2}\cdot 2\cos(xy+2z).$$

Therefore,

$$\begin{aligned}\nabla f(x,y,z) = &\left[\frac{y(x^2-y^2)\cos(xy+2z) - 2x\sin(xy+2z)}{(x^2-y^2)^2}\right]\hat{\mathbf{x}}\\ &+\left[\frac{x(x^2-y^2)\cos(xy+2z) + 2y\sin(xy+2z)}{(x^2-y^2)^2}\right]\hat{\mathbf{y}}\\ &+\left[\frac{2\cos(xy+2z)}{x^2-y^2}\right]\hat{\mathbf{z}}.\end{aligned}$$

Question 5: Find the size (modulus of) the gradient and provide a normalized vector in the direction of steepest increase of the function $g(x, y) = 3x + y^2/x$ at the point $x = 1$, $y = 1$, given that x and y are Cartesian coordinates.

The gradient of $g(x, y)$ is given by

$$\nabla g(x, y) = \left(3 - \frac{y^2}{x^2}\right)\hat{\mathbf{x}} + \left(2\frac{y}{x}\right)\hat{\mathbf{y}}.$$

Hence at $x = 1$, $y = 1$, the gradient $\nabla g(1, 1) = 2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$. It might be easier to proceed with the vector form, *i.e.*,

$$\nabla g = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The modulus of the gradient is therefore

$$|\nabla g| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

This enables us to normalise the vector,

$$\frac{\nabla g}{|\nabla g|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

This unit vector tells us such a direction would result in the change of a scalar field $g(x, y)$ the most rapidly (the steepest ascent) along x and y (which is the physical interpretation of the gradient ∇).