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Signals and Control - Control

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Special thanks to *Haroon Chughtai*. His notes greatly enlightened me when preparing this new set.

Errata

Colors are used to code different actions applied to the context. blue is used for replacement, red for removal, and green for addition.

8^{th} Mar 2024

- *Section 2, Example 2.1* Two typos have been corrected. On page 8:

$$y(t) = g(t) = \mathcal{L}^{-1}[G(s)] = -e^{-t/2} + -e^{-t/3}$$

$$y(t) = 2e^{-t/2} - 3e^{-t/3} + 1$$

- *Section 6.6* A typo has been corrected. On page 31:

  The phase plot at $-90°$ at $r = r_c = 1$.

16^{th} Mar 2021

- *Section 3.1.1* Added the missing bracket in the expression of $b_i^{(k)}$.
- *Section 3.2* The typos $t \to 0$ in the expression of final value theorem and the example below has been fixed.
- *Section 6.3* Fixed the wrong legend. The legend in the Bode plot of $(\frac{1}{s})^n$ should start from $n = 0$.
- *Section 8.2.4* Fixed the typo in the example. The second solution should be k_2.

14^{th} Mar 2021

- *Section 1.1 & 1.2* I have changed the bullet points to tables.
- *Section 8.1* The typo in the transfer function has been fixed.

Source File

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Contents

1 Laplace Transform

Fourier Transform:

$$F(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\mathrm{d}t$$

Laplace Transform:

$$F(s) = \mathcal{L}(f(t)) = \int_{-\infty}^{+\infty} f(t)e^{-st}\mathrm{d}t$$

where $s = \sigma + j\omega$.

- **Reason we need Laplace transform:**
  - Laplace transform is the generalization of Fourier transform. It is defined for a larger class of functions than Fourier transform.
  - This is because s can be defined anywhere in the complex plane.
- In control, we usually take the lower boundary of the integral to 0, rather than $-\infty$. We call this **unilateral Laplace transform**.

$$F(s) = \mathcal{L}(f(t)) = \int_{0}^{\infty} f(t)e^{-st}\mathrm{d}t$$

Example 1.1

Fourier transform does not exist for $f(t) = e^t$, this is because $f(t)$ is not convergent.

However, we can find the Laplace transform of $f(t) = e^t$.

$$F(s) = \mathcal{L}(e^t) = F(s) = \mathcal{L}(f(t)) = \int_{0}^{\infty} e^{(1-s)t}\mathrm{d}t = -\frac{1}{1-s}$$

This provided that

$$\lim_{t\to\infty} e^{(1-s)t} = 0$$

which is true for s such that $\Re(s) > 1$.

- We make the signal **converge** ($e^{-\sigma t} \cdot f(t)$ converges) by multiplying $e^{-\sigma t}$, so that it can be analysable.

$$e^{-st} = e^{-(\sigma+j\omega)t} = e^{-\sigma t} \cdot e^{-j\omega t}$$

1.1 Table of Laplace Transform

Table 1: Table of Laplace transform

	$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
Time delay	$\delta(t-\tau)$	$e^{-\tau s}$
Impulse	$\delta(t)$	1
Step	$u(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(\omega_0 t)$	$\frac{\omega}{s^2+\omega_0^2}$
Cosine	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$

Derivation 1.1: Time delay

Let $g(t) = f(t-T)$, take Laplace transform

$$\begin{aligned} G(s) &= \int_0^{+\infty} e^{-st} g(t)\mathrm{d}t = \int_0^{+\infty} e^{-st} f(t-T)\mathrm{d}t \\ &= \int_0^{+\infty} e^{-s(\tau+T)} f(\tau)\mathrm{d}\tau = e^{-sT} \underbrace{\int_0^{+\infty} e^{-s\tau} f(\tau)\mathrm{d}\tau}_{F(s)} \\ &= e^{-sT} F(s) \end{aligned}$$

Example 1.2

Evaluate $\mathcal{L}[\sin(2t-3)]$.

Let $f(t) = \sin(2t)$, take Laplace transform: $F(s) = \frac{2}{s^2+4}$.

Due to $\delta(t-\tau) \xrightarrow{\mathcal{L}} e^{-\tau s}$

$$\mathcal{L}[\sin(2t-3)] = \mathcal{L}[\sin 2(t-\frac{3}{2})] = e^{-\frac{3}{2}s}\frac{2}{s^2+4}$$

1.2 Properties of Laplace Transform

Table 2: Properties of Laplace transform

	$f(t)$	$F(s)$
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
Differentiation	$f'(t)$	$sF(s) - f(0)$ $s^n F(s) - s^{n-1} f(0) - ... - f^{n-1}(0)$
Integration	$\int_0^t f(\tau)\mathrm{d}\tau$	$\frac{1}{s}F(s)$
Convolution	$f(t) * g(t)$	$F(s)G(s)$
Exponential scaling	$e^{-at}f(t)$	$F(s+a)$
Time scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
Time shifting	$f(t-\tau)u(t-\tau)$	$e^{-\tau s}F(s)$

Derivation 1.2: Exponential scaling

Let $g(t) = e^{-at}f(t)$, take Laplace transform:

$$\begin{aligned} G(s) &= \int_0^\infty e^{-st} g(t)\mathrm{d}t \\ &= \int_0^\infty e^{-st} e^{-at} f(t)\mathrm{d}t \\ &= \int_0^\infty e^{-(s+a)t} f(t)\mathrm{d}t \\ &= F(s+a) \end{aligned}$$

Example 1.3

Evaluate $\mathcal{L}[e^{-t}\sin(2t)]$.

Let $f(t) = \sin(2t)$, take Laplace transform: $F(s) = \dfrac{2}{s^2+4}$.

Due to $e^{-at}f(t) \overset{\mathcal{L}}{\rightarrow} F(s+a)$

$$\mathcal{L}[e^{-t}\sin(2t)] = F(s+1) = \frac{2}{(s+1)^2+4}$$

2 ODE description of LTI systems

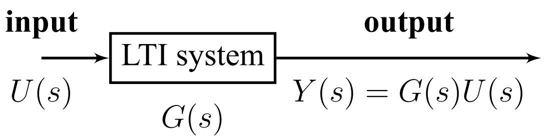


Fig. 1: *Linear time-invariant system*

- LTI systems can be described by ODEs with constant coefficients in time domain:

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + ... + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + ... + b_1 \frac{\mathrm{d}u(t)}{\mathrm{d}t} + b_0 u(t)$$

- Assuming there is zero initial conditions, take Laplace transform:

$$a_n s^n Y(s) + ... + a_1 s Y(s) + a_0 Y(s) = b_m s^m U(s) + ... + b_1 s U(s) + b_0 U(s)$$

  where $\mathcal{L}[y(t)] = Y(s)$ and $\mathcal{L}[u(t)] = U(s)$.

- **Transfer Function** is defined as:

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + ... + b_1 s + b_0}{a_n s^n + ... + a_1 s + a_0}, \quad m \le n}$$

  - Laplace transform of **output** divided by Laplace transform of **input**, assuming zero initial conditions.
  - Ratio of polynomial in s.
  - Transfer function is independent of the form of the input

Properties of transfer functions:

- Linear functions

$$G(s)(aU_1(s) + bU_2(s)) = aG(s)U_1(s) + bG(s)U_2(s)$$

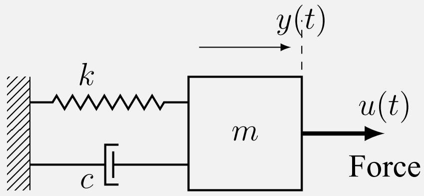
- Commutative

$$G_1(s)G_2(s) = G_2(s)G_1(s)$$

- Associative

$$G_1(s) + G_2(s) = G_2(s) + G_1(s)$$

Example 2.1 Mass-Spring-Damper system



A Mass-Spring-Damper system can be characterized by ODE:

$$m\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ky(t) = u(t)$$

Take Laplace transform with zero initial condition:

$$ms^2Y(s) + csY(s) + kY(s) = (ms^2 + cs + k)Y(s) = U(s)$$

So the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + cs + k}$$

If $m = 6$, $c = 5$ and $k = 1$,

- to find the impulse response:

$$Y(s) = G(s)U(s) = G(s) = \frac{1}{6s^2 + 5s + 1} = \frac{-2}{2s+1} + \frac{3}{3s+1} = \frac{-1}{s+\frac{1}{2}} + \frac{1}{s+\frac{1}{3}}$$

$$y(t) = g(t) = \mathcal{L}^{-1}[G(s)] = -e^{-t/2} + e^{-t/3}$$

- to find the step response:

$$Y(s) = G(s)U(s) = \frac{1}{s}\frac{1}{6s^2 + 5s + 1} = \frac{4}{2s+1} - \frac{9}{3s+1} + \frac{1}{s} = \frac{2}{s+\frac{1}{2}} - \frac{3}{s+\frac{1}{3}} + \frac{1}{s}$$

$$y(t) = 2e^{-t/2} - 3e^{-t/3} + 1$$

Comments:

- From the example above, the solutions of denominator of the transfer function are $s = -\frac{1}{2}$ and $s = -\frac{1}{3}$;
- These two solutions are the **poles** of the system.
- Poles define the long-time behaviours of the system: whether the system can stay at certain level, or converges to 0 or ∞.
- $s = -\frac{1}{2}$ and $s = -\frac{1}{3}$ are real, negative solutions, so as $t \to \infty$, for step response, $y(t) \to 1$, (and for impulse response, $y(t) \to 0$). They have no affect on long-term behaviour of the system.

2.1 Zero-pole-gain

Transfer function can be written as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + ... + b_1 s + b_0}{a_n s^n + ... + a_1 s + a_0}$$
$$= \boxed{K\frac{(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}}$$

where K is the gain.

- Zeros $\{z_1,\ z_2,\ ...,\ z_m\}$ are the solutions of $N(s) = 0$;
- Poles $\{p_1,\ p_2,\ ...,\ p_n\}$ are the solutions of $D(s) = 0$;

Example 2.2

Find the poles and zeros of a LTI system

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 2\frac{\mathrm{d}u}{\mathrm{d}t} + u$$

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1. Obtain the transfer function by Laplace transform

$$s^2 Y(s) + 5sY(s) + 6Y(s) = 2sU(s) + U(s)$$

   This gives

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+5s+6}$$

2. Factorize the transfer function
   - $2s + 1 = 0$, zero at $s = -\frac{1}{2}$.
   - $(s+3)(s+2) = 0$, poles at $s = -2$ and $s = -3$

Example 2.3

Find the ODE representing the system with

- poles at $-1 \pm 2j$
- zero at -4
- gain factor 3

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1. Find the transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = 3\frac{s-(-4)}{[s-(-1-2j)][s-(-1+2j)]} = 3\frac{s+4}{s^2+2s+5}$$

2. ODE can be found by taking Inverse Laplace transform:

$$(s^2+2s+5)Y(s) = 3(s+4)U(s) \quad \xrightarrow{\mathcal{L}} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 3\frac{\mathrm{d}u}{\mathrm{d}t} + 12u$$

2.2 Block Diagrams

Block diagrams are used to visually represent LTI systems.

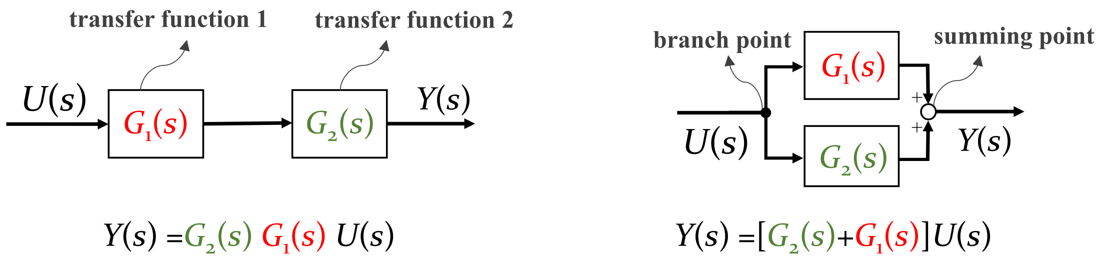
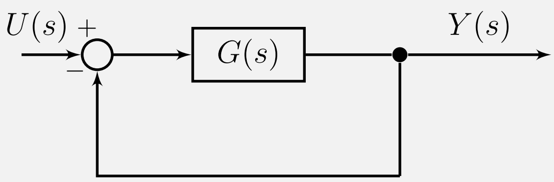


Fig. 2: *Block diagram explanation*

Example 2.4

Consider the simplest negative feedback system, as shown below. Find the transfer function of the system.



Note: the minus sign in the figure indicates the negative feedback in the system.

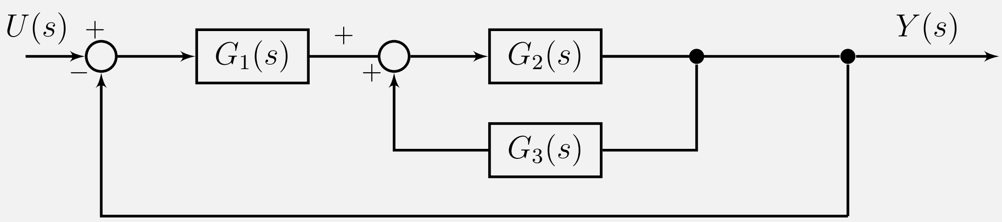
$$Y(s) = G(s)(U(s) - Y(s))$$

Rearrange, then we find the transfer function of this closed-loop system:

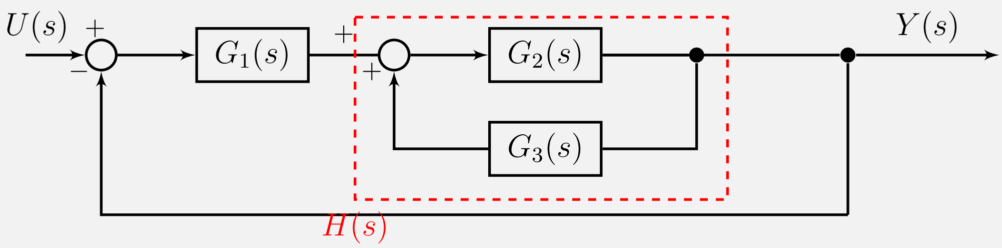
$$\boxed{\frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)}}$$

Example 2.5

Derive the transfer function of the following negative feedback system.



Consider the dashed section:



The transfer function of the dashed section, $H(s)$, that is equivalent to the combination of $G_2(s)$ and $G_3(s)$.

- Let the input to the dashed section be $X(s)$:

$$Y(s) = G_2(s)(X + G_3(s)Y(s))$$

- Rearrange:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G_2(s)}{1 - G_2(s)G_3(s)}$$

Then we are going to find the equivalent transfer function to the whole system:

$$Y(s) = (U(s) - Y(s))G_1(s)H(s)$$

This gives us the final result:

$$T.F. = \frac{Y(s)}{U(s)} = \frac{G_1(s)H(s)}{1 + G_1(s)H(s)}, \quad \text{where } H(s) = \frac{G_2(s)}{1 - G_2(s)G_3(s)}$$

3 Poles and Stability

Poles are the solutions of the characteristic equation(the denominator of a transfer function) which defines the stability of the system.

The impulse response have different look when their poles have different locations in the s-domain.

- **Poles at:** $s = \sigma \pm j\omega$
- **Impulse response:**

$$g(t) = Ae^{(\sigma \pm j\omega)t} = Ae^{\sigma t}(\cos(\omega t) \pm j\sin(\omega t))$$

where ω is the frequency of the oscillation.

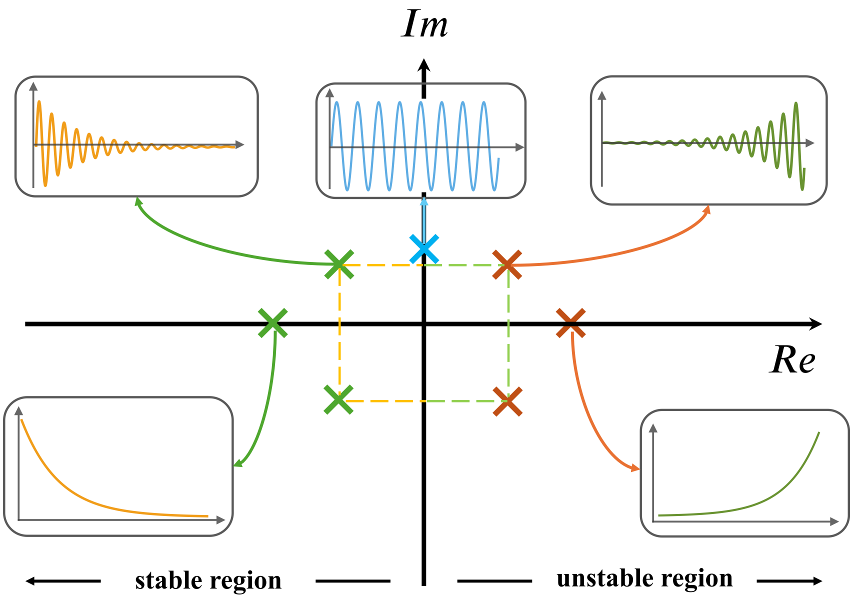


Fig. 3: *Poles located in the left half-plane are stable while poles located in the right half-plane are not stable.*

Stability is evaluated by the signs of the real parts of the poles:

- **Asymptotic stability**: when $\Re\{p_i\} < 0$ for all poles p_i: (Fig. 3 - stable region)
  - The output decays within an exponential envelope approaching asymptotically 0.
- **Instability** when at least one pole is $\Re\{p_i\} > 0$: (Fig. 3 - unstable region)
  - The output grows without bound.
- **Marginal stability** when $\Re\{p_i\} = 0$ for some poles and $\Re\{p_i\} < 0$ for all other poles: (Fig. 3 - middle)
  - The output never decays or grows in amplitude, and shows sustained oscillations.

3.1 Finding Stability

We have 2 methods to find the stability of the system:

1. **Solving the characteristic equation** to find poles. The signs of the real part of the poles indicates the stability of the system.
2. **Routh stability criterion**. This is particularly useful if the characteristic equation is of high order and tedious to solve.

3.1.1 Routh Stability Criterion

A Routh table can be used to determine the number of poles with positive real parts in a transfer function. The key concepts are

- we only care about the denominator, $D(s)$, of the transfer function;
- **number of change of sign in the 1st column = number of poles with positive real parts**.

To create a Routh table:

- Given a transfer function:

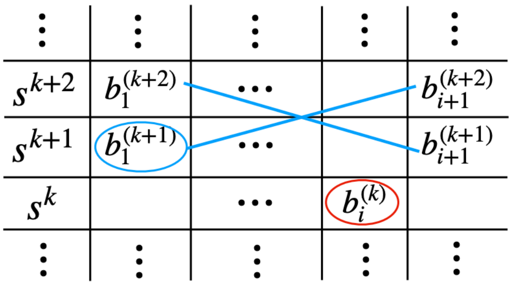
$$G(s) = \frac{N(s)}{D(s)} \quad \text{where} \quad D(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0.$$

- Fill in the corresponding values to the Routh Table:

s^n	a^n	a^{n-2}	a^{n-4}
s^{n-1}	a^{n-1}	a^{n-3}	a^{n-5}
s^{n-2}	$b_1^{(n-2)}$	$b_2^{(n-2)}$	$b_3^{(n-2)}$
s^{n-3}	$b_1^{(n-3)}$	$b_2^{(n-3)}$	$b_3^{(n-3)}$
$\vdots$	$\vdots$	$\vdots$	
s^2	$b_1^{(2)}$	$b_2^{(2)}$	0
s^1	$b_1^{(1)}$	0	
s^0	$b_1^{(0)}$		

where

$$b_i^{(k)} = \frac{b_1^{(k+1)} \times b_{i+1}^{(k+2)} - b_1^{(k+2)} \times b_{i+1}^{(k+1)}}{b_1^{(k+1)}}$$



- The system is asymptotically stable *if and only if*
  - $a_i > 0, \ \forall i$ (for all i values)
  - **no change of sign** in the 1st column of the Routh Table.

Example 3.1

Given the transfer function $G(s) = \dfrac{2s+1}{s^2+5s+6}$, evaluate the stability of the system.

s^2	1	6
s^1	5	0
s^0	$\frac{6\times5-1\times0}{5} = 6$	

No change of sign in the 1st column since $1 \to 5 \to 6$. Therefore, it has 0 unstable poles.

Note: for the cells with no value from the characteristic function, but this cell is involved in calculation, fill in 0 instead.

Example 3.2

Given the transfer function $G(s) = \frac{2s+1}{s^3+s^2+3s+10}$, evaluate the poles.

s^3	1	3	0
s^2	1	10	0
s^1	$\frac{3\times-1\times10}{1} = -7$	(*assume*) x(=0)	
s^0	$\frac{-7\times10-1\times x}{-7} = 10$		

To find the value of x:

$$x = \frac{10\times0-3\times0}{1} = 0$$

It has 2 unstable poles since the sign has changed twice($1 \to 1 \to -7 \to 10$) in the first column.

Note: add a column to the right when necessary.

3.2 Final Value Theorem

Final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to0} sY(s)$$

where

$$y(t) = C_1e^{p_1t} + C_2e^{p_2t} + \ldots + C_ne^{p_nt}$$

and

$$Y(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}$$

- **Constant** when $p_1 = 0$ and all other p_i have $\Re(p_i) < 0$. Then the final value is C_1.
- **Undefined** when there are p_i on the imaginary axis. Then $y(t)$ oscillates and does not converge.
- **Unbounded** when there are any p_i with $\Re(p_i) > 0$.

Example 3.3

Find the steady state of the step response for the system $G(s) = \frac{3}{s-2}$.

If we apply the final value theorem *blindly*:

$$\lim_{t\to\infty} y(t) = \lim_{s\to0} sY(s) = \lim_{s\to0} G(s) = -\frac{3}{2}$$

This is a wrong answer!!

However, we know that the step response is

$$Y(s) = U(s)G(s) = \frac{3}{s-2}\frac{1}{s} = \frac{-1.5}{s} + \frac{1.5}{s-2}$$

Take the inverse Laplace's transform:

$$y(t) = -\frac{3}{2} + \frac{3}{2}e^{2t}$$

Therefore

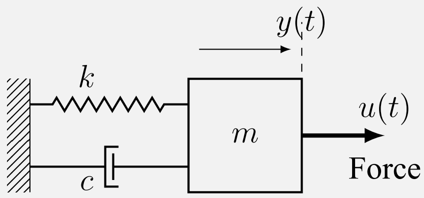
$$\lim_{t \to \infty} y(t) = \infty$$

Unbounded, since $\Re(p_i) > 0$ holds for any p_i. **This is the correct answer!!**

4 Time Response Analysis

Higher order systems can be analysed by approximating them as 1st or 2nd order systems.

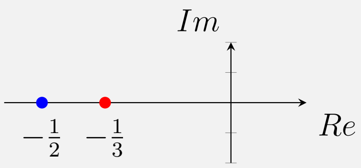
Example 4.1 Mass-Spring-Damper system



Impulse response: $g(t) = -e^{-\frac{1}{2}t} + e^{-\frac{1}{3}t}$ with poles at $s = -\frac{1}{2}$, $s = -\frac{1}{3}$.

Recall that: The poles determine how quickly the system moves towards the steady state. Since $e^{-\frac{1}{2}t}$ decays faster than $e^{-\frac{1}{3}t}$, $s = -\frac{1}{3}$ is the dominant pole.

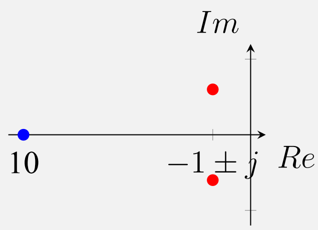
Dominant pole lies closer to the imaginary axis in s-plane.



Example 4.2

Obtain the dominant pole approximation of $G(s) = \dfrac{10}{(s+10)(s^2+2s+2)}$.

Poles $s = -10$ and $s = -1 \pm j$, their locations are shown in the s-plane below. Since $s = -1 \pm j$ lie closer to the imaginary axis, these are Dominant poles of the system.



We can approximate the system by

- keeping only the dominant poles
- ensuring the equal steady state for unit step response

For the case above, since $s = -1 \pm j$ are the dominant poles:

$$G(s) = \frac{10}{(s+10)(s^2+2s+2)} \quad \rightarrow \quad \hat{G}(s) = \frac{K}{s^2+2s+2}$$

To determine the value of K:

$$\begin{cases} y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = \frac{1}{2} \\ \hat{y}(\infty) = \lim_{s\to 0} s\hat{Y}(s) = \lim_{s\to 0} s\hat{G}(s)\frac{1}{s} = \frac{K}{2} \end{cases}$$

$\therefore \quad K = 1.$

4.1 Good and Bad Approximations

- If the dominant poles are very different from the other poles, this will lead to a good approximation;

Example 4.3

The system $G(s) = \dfrac{10}{(s+10)(s^2+2s+2)}$ has poles at $s = -10$, $s = -1 \pm j$.

Since poles are not close each other, the 2nd order approximation is almost equivalent to the original system.

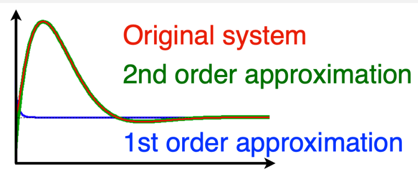


Fig. 4: *A good approximation,* $G(s) = \frac{10}{(s+10)(s^2+2s+2)}$

- If the dominant pole is close to the other poles, the approximation will be imprecise.

Example 4.4

The system $G(s) = \dfrac{1}{(s+51)(s^2+100s+9000)}$ has poles at $s = -51$, $s = -50 \pm 81j$.

In this case, poles are close to each other, which leads to the bad approximation.

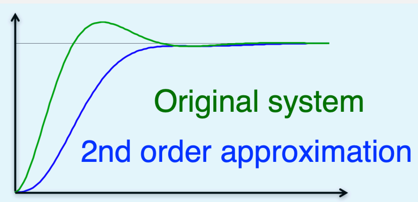


Fig. 5: *A bad approximation,* $G(s) = \frac{1}{(s+51)(s^2+100s+9000)}$

4.2 Time Response Analysis for 1st Order Systems

$$T\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = Ku(t) \quad \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\longleftrightarrow}} \quad (Ts+1)Y(s) = KU(s)$$

- The transfer function for a 1st order system is $G(s) = \frac{K}{Ts+1}$ with poles at $s = -\frac{1}{T}$, T is known as the **time constant**.
- Time constant determines how fast the system moves towards the steady state.

4.2.1 Impulse Response of 1st Order Systems

$$U(s) = 1 \quad \longrightarrow \quad \boxed{G(s) = \frac{K}{Ts+1}} \quad \longrightarrow \quad Y(s) = \frac{K}{Ts+1} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \frac{K}{T}e^{-\frac{1}{T}t}$$

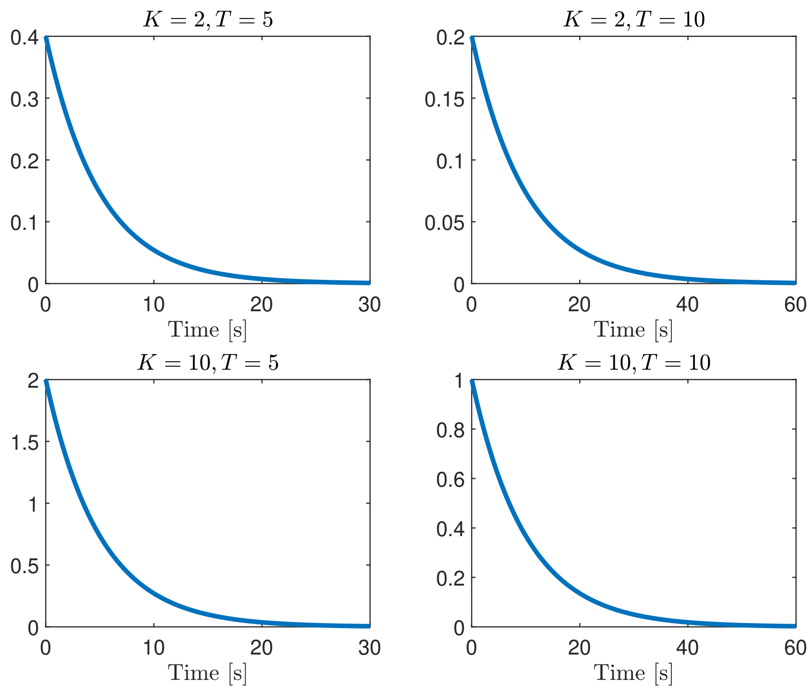


Fig. 6: *Impulse response of 1st order systems*

4.2.2 Step Response of 1st Order Systems

$$U(s) = \frac{1}{s} \quad \longrightarrow \quad \boxed{G(s) = \frac{K}{Ts+1}} \quad \longrightarrow \quad Y(s) = \frac{1}{s}\frac{K}{Ts+1} \quad \xrightarrow{\mathcal{L}^{-1}} \quad y(t) = K(1 - e^{-\frac{1}{T}t})$$

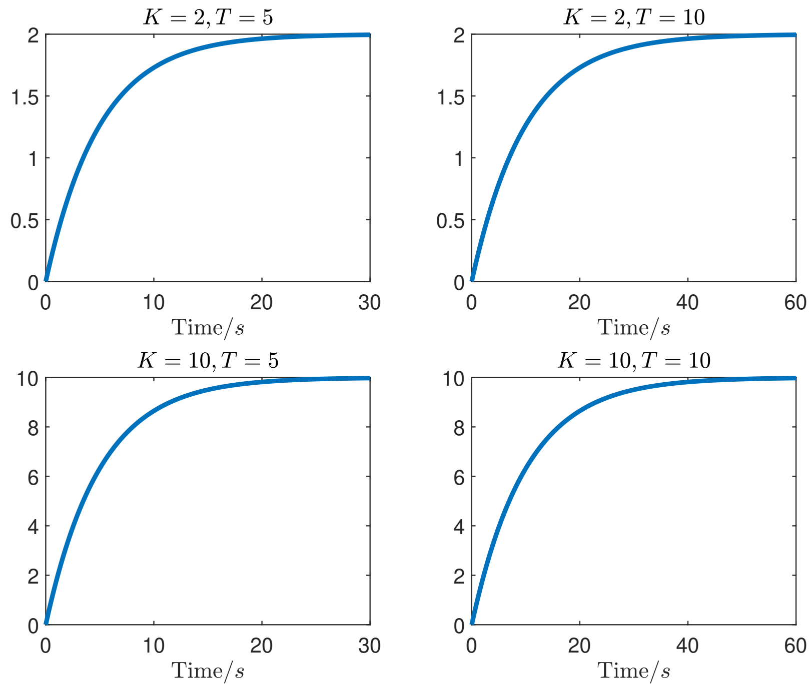


Fig. 7: *Step response of 1st order systems*

4.3 Time Response Analysis for 2nd Order Systems

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 2\zeta\omega_n \frac{\mathrm{d}y(t)}{\mathrm{d}t} + \omega_n^2 y(t) = K\omega_n^2 u(t)$$

The transfer function for a 2^{nd} order system is $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, where

- ζ is **damping ratio**. It measures how much the system oscillates as the output decays towards the steady state.
- ω_n is **undamped natural frequency**. It measures how fast the system oscillates during the transient response.

Time response, $Y(s)$, of 2^{nd} order systems:

- Impulse response:

$$Y(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Step response:

$$Y(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}$$

The characteristic equations are both $s^2+2\zeta\omega_n s+\omega_n^2 = 0$, with poles at $s = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$. This gives 3 cases:

$$s = \begin{cases} s = \pm j\omega_n \ , & \zeta = 0 \\ s = (-\zeta \pm j\sqrt{1-\zeta^2})\omega_n \ , & 0 < \zeta < 1 \\ s = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n \ , & \zeta > 1 \end{cases}$$

4.3.1 Case 1: $\zeta = 0$, Undamped Response

Undamped response happens when damping ratio $\zeta = 0$, with poles at $s = \pm j\omega_n$. The undamped step response is **purely sinusoidal**:

$$Y(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} \xrightarrow{\mathcal{L}^{-1}} y(t) = K(1 - \cos(\omega_n t))$$

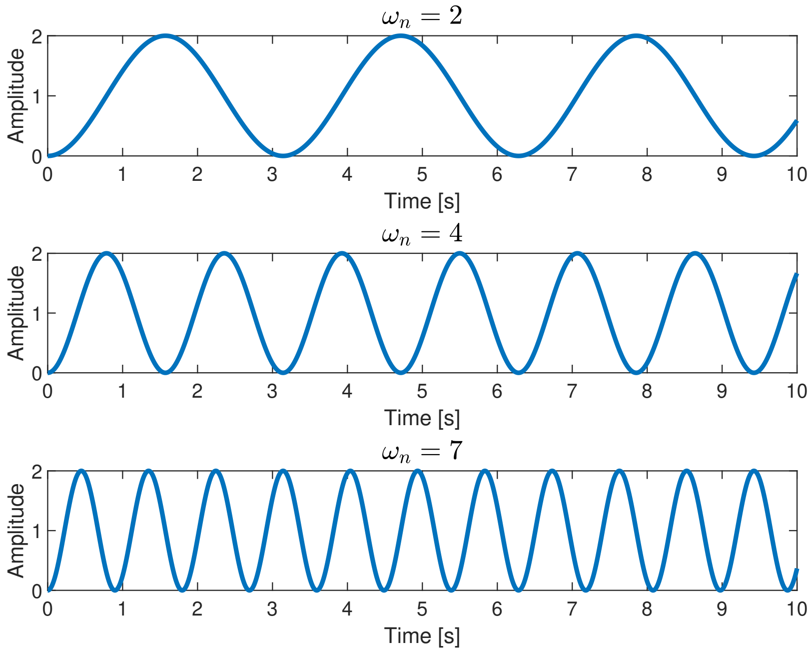


Fig. 8: *Undamped response of 2nd order systems.* $K = 1$

4.3.2 Case 2: $0 < \zeta < 1$, Underdamped Response

Underdamped response happens when damping ratio $0 < \zeta < 1$, with 2 conjugate poles at $s = (-\zeta \pm j\sqrt{1-\zeta^2})\omega_n$.

The underdamped step response is **the combination of exponential function and sinusoidal function**:

- Step response:

$$Y(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = K(\frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2})$$

- Take inverse Laplace transform:

$$\xrightarrow{\mathcal{L}^{-1}} y(t) = K\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})\right]$$

  – $\zeta\omega_n$ is known as **decay time constant**;

– ω_d is known as **damped natural frequency** with the expression:

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

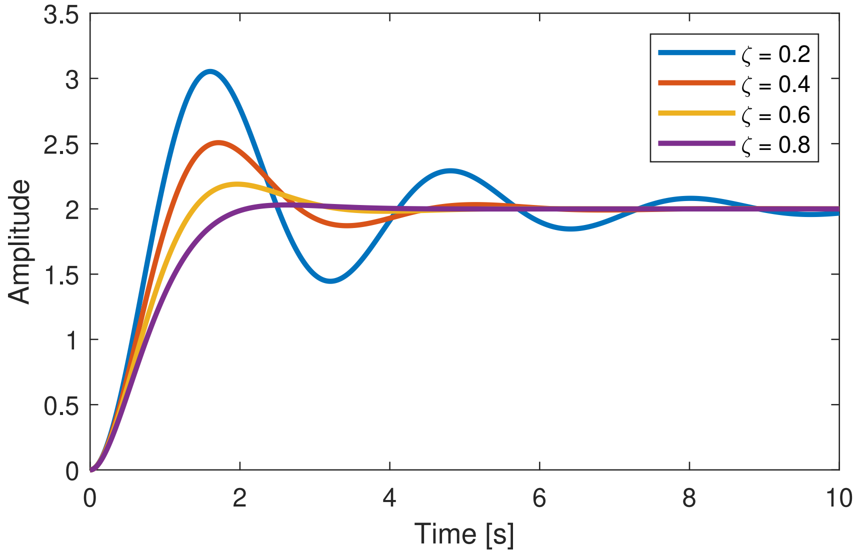


Fig. 9: *Underdamped response of 2nd order systems.* $\omega_n = 2$, $K = 2$

4.3.3 Case 3: $\zeta > 1$, Overdamped Response

Overdamped response happens when damping ratio $\zeta > 1$, with 2 negative, real poles at $s = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n$ which are refer as α and β.

The overdamped step response **quickly converges to a stable value**:

- Step response:

$$Y(s) = \frac{K\omega_n^2}{(s-\alpha)(s-\beta)}\frac{1}{s} = \frac{K\alpha\beta}{s(s-\alpha)(s-\beta)} = K\left[\frac{1}{s} + \frac{1}{\alpha-\beta}(\frac{\beta}{s-\alpha} - \frac{\alpha}{s-\beta})\right]$$

- Take inverse Laplace transform:

$$\xrightarrow{\mathcal{L}^{-1}} \quad y(t) = K\left(1 + \frac{\beta e^{\alpha t} - \alpha e^{\beta t}}{2\omega_n\sqrt{\zeta^2-1}}\right)$$

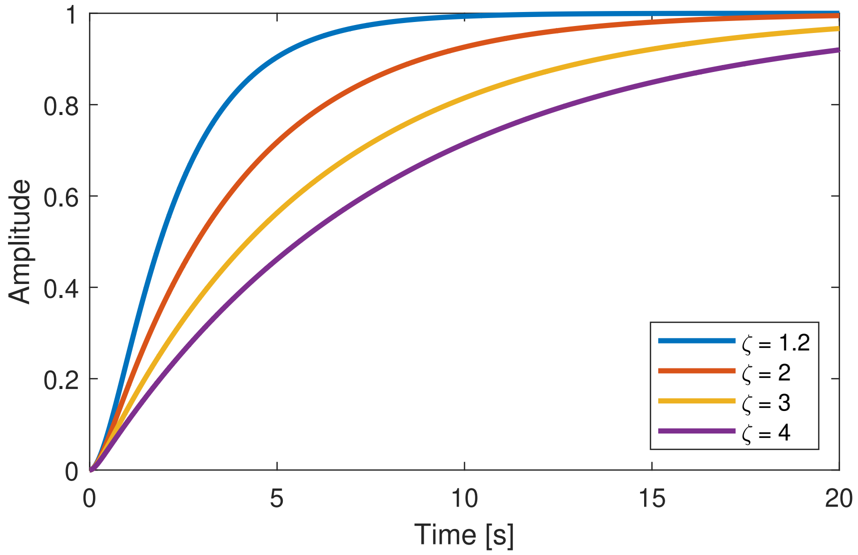


Fig. 10: *Overdamped response of 2nd order systems.* $\omega_n = 1,\ K = 1$

4.4 Transient Specification of 2nd Order Systems

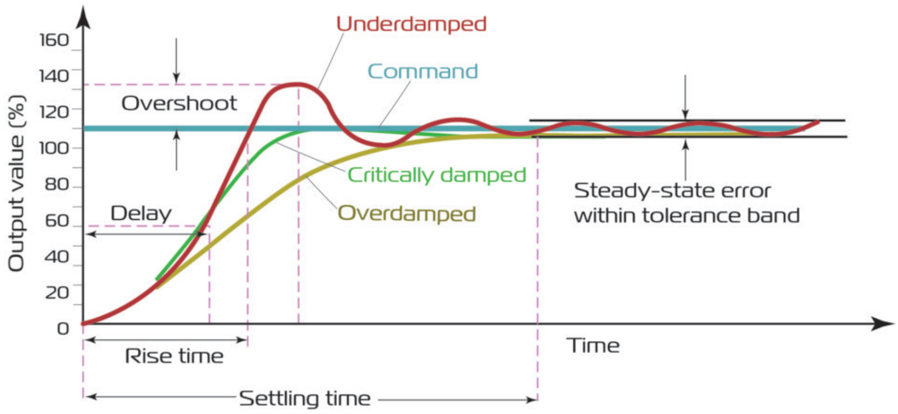


Fig. 11: *Overshoot, Rise time, Setting time and Steady-state error* [1]

- **Rise time, t_r:** the time taken for the output to go from 10% to 90% of the final value.

$$t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

- **Peak time, t_p:** the time taken for the output to reach its maximum value.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

[1] Figure adopted from https://www.motioncontroltips.com/how-to-address-overshoot-in-servo-control/

- **2% Settling time, $t_{s2\%}$:** the time taken for the signal to be bounded to within a tolerance of x% (here, 2%) of the steady state value.

$$t_{s2\%} = \frac{4}{\zeta\omega_n}$$

- **Overshoot**:

$$\textbf{overshoot} = \frac{\text{max value - min value}}{\text{final value}} \times 100$$

- **Steady-state error**: the difference between the input step value and the final value.

5 PID Control

5.1 Open-loop Control

- No feedback in open-loop control, output has no effect on the control action;
- Open-loop control is convenient when it is hard to measure output;
- It is only used if the uncontrolled plant dynamics is **perfectly known** and is not subject to any environmental changes.

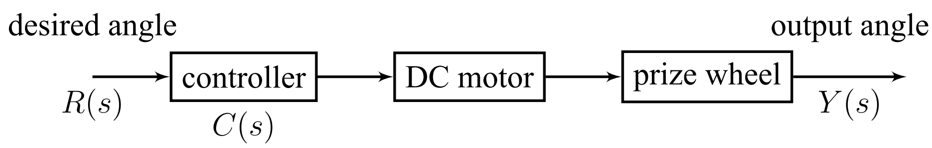


Fig. 12: *Open-loop control system for a prize wheel*

If we want the output follows the desired input, *i.e.* $Y(s) = R(s)$, this is achievable by choosing the controller as $C(s) = P^{-1}(s)$.

5.2 On-off (bang-bang) Control

- On-off control is the simplest closed-loop feedback control
- On-off control may cause oscillating errors due to the frequent switchings.
- Potential high input to the plant and wear-and-tear of the system.

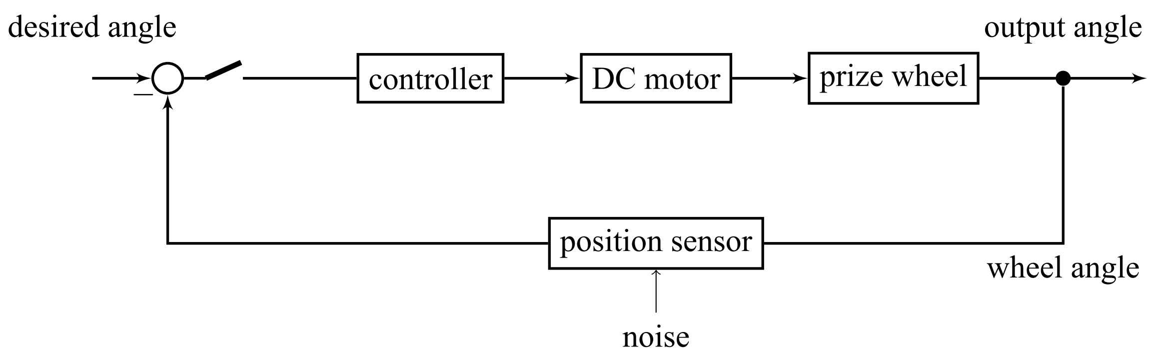


Fig. 13: *On-off control system for a prize wheel*

Take the example of the prize wheel. Position sensor measures the output angle, and compare this angle with the desired angle. Then use the switch*(on-off)* to control the strength of the DC motor.

5.3 Gain (proportional) Control

- Gain control applies control input that is proportional to the current error.
- It has very quick reaction to the current error
- But it is possible to **overshoot** when the error is magnified too much.
- Also, gain control leads to a non-zero **steady-state error**.

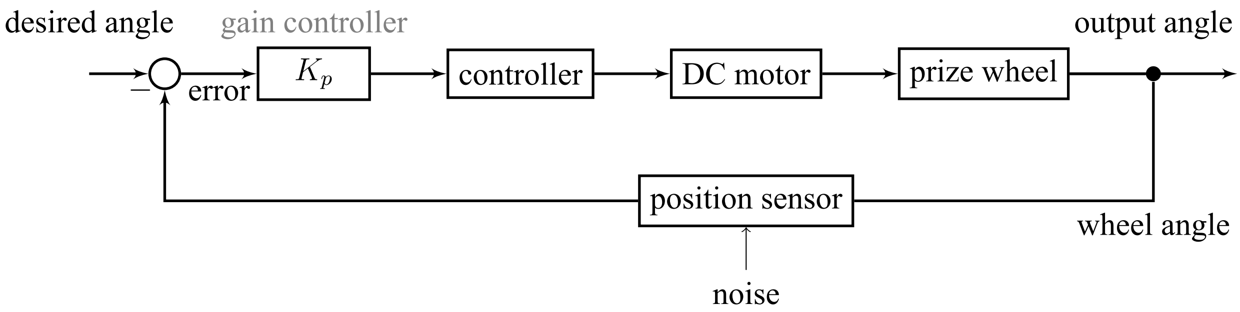


Fig. 14: *Gain control system for a prize wheel*

Take the example of the prize wheel. The position sensor measures the wheel angle and compare it with the desired angle. The error is then calculated and sent into the gain controller. The gain controller magnifies the error and send to the DC motor for further action.

5.4 PID Control

PID control is the most common controller used in industry which aims to reduce the tracking error between desired and measured output of the system.

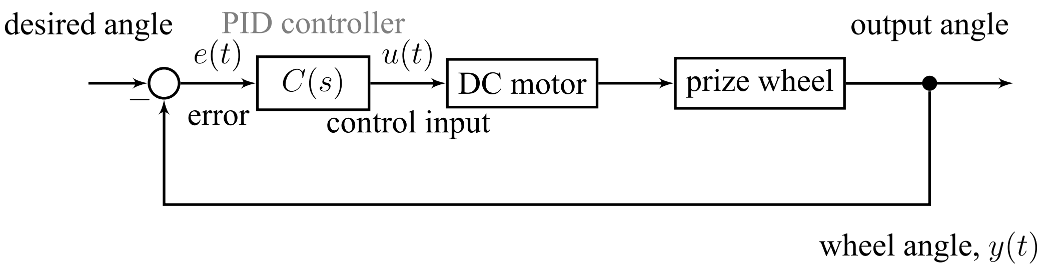


Fig. 15: *PID control system for a prize wheel*

- Control input is the sum of **Proportional**, **Integral** and **Derivative** of the error.
- Control input can be represented as:

$$u(t) = K_P\, e(t) + K_I \int_0^t e(\tau)\mathrm{d}\tau + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

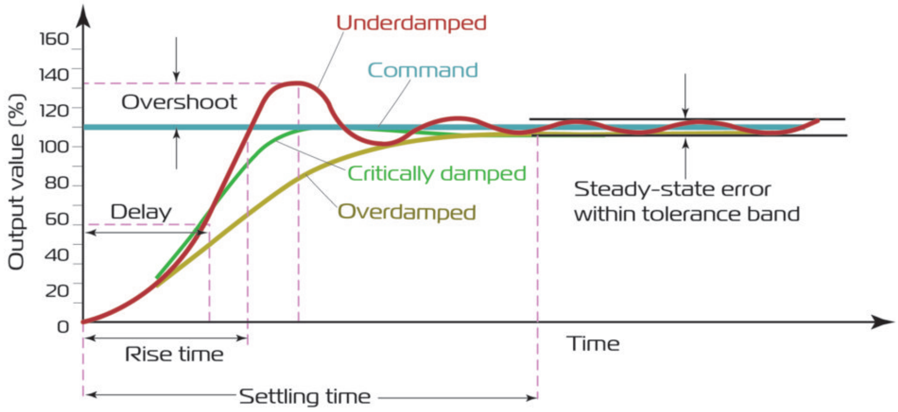
- PID controller can be represented as:

$$C(s) = K_P + \frac{K_I}{s} + K_D s$$

– K_P stands for **proportional gain**. It uses the information from the *present error*.
– K_I stands for **integral gain**. It uses the information from the *past error*. It removes the steady-state error.
– K_P stands for **derivative gain**. It uses the *future error*. It improves the stability but has no effect on steady-state error.

5.4.1 System response to K_P, K_I and K_D

Increase of	Overshoot	Rise Time	Settling Time	Steady-state error
K_P	Increase	Decrease	Small increase	Decrease
K_I	Increase	Decrease	Increase	Eliminate
K_D	Decrease	Decrease	Decrease	No impact



6 Bode Plots

Bode plots are used to evaluate the behaviour of closed-loop systems.

- **Gain plot**

$$20\log_{10}|G(j\omega)| \text{ [in dB]} \quad \text{v.s.} \quad \omega \text{ (in } \log_{10} \text{ scale)}$$

- **Phase plot**

$$\angle G(j\omega) \text{ [in deg]} \quad \text{v.s.} \quad \omega \text{ (in } \log_{10} \text{ scale)}$$

The Bode plots of a complex system can be obtained by the addition of Bode plots for simple systems.

- For gain plot:

$$20\log_{10}|G(j\omega)| = 20\log_{10}|G_1(j\omega)| + 20\log_{10}|G_2(j\omega)|$$

- For phase plot:

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega)$$

6.1 Bode Plots for Constant Gain, $G(s) = K$

- **Gain plot**: $20\log_{10}|G(j\omega)| = 20\log_{10} K$
- **Phase plot**: $\angle G(j\omega) = 0°$

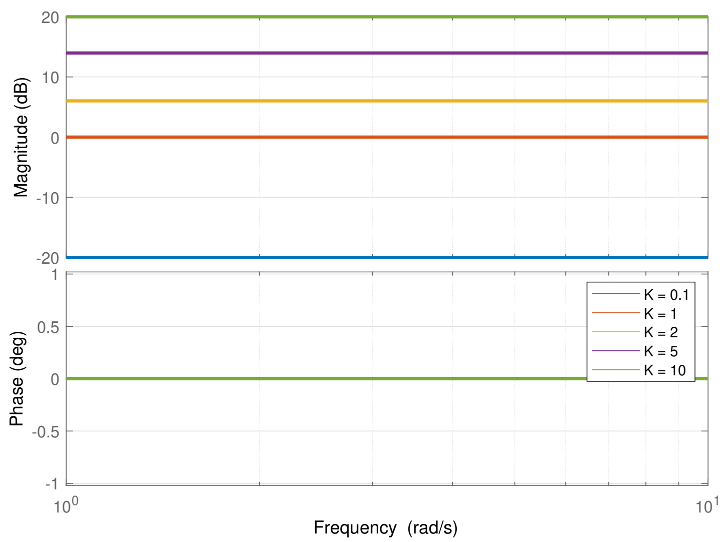


Fig. 16: *Bode plot of $G(s) = K$. Magnitude plot is a constant.*

6.2 Bode Plots for Differentiator, $G(s) = s^n$

- **Gain plot**:

$$20\log_{10}|G(j\omega)| = 20\log_{10}|(j\omega)^n| = n \cdot 20\log_{10}|(j\omega)| = \boxed{n \cdot 20\log_{10}\omega}$$

- **Phase plot**: $\angle G(j\omega) = \angle(j\omega)^n = \boxed{n \cdot 90°}$
- Gain difference Δ over the interval $[\omega_1, \omega_2]$:

$$\Delta = n \cdot 20(\log_{10}\omega_1 - \log_{10}\omega_2) = \boxed{n \cdot 20\log_{10}\frac{\omega_1}{\omega_2}}$$

This gives the slope$= n \cdot 20$ [dB/decade]

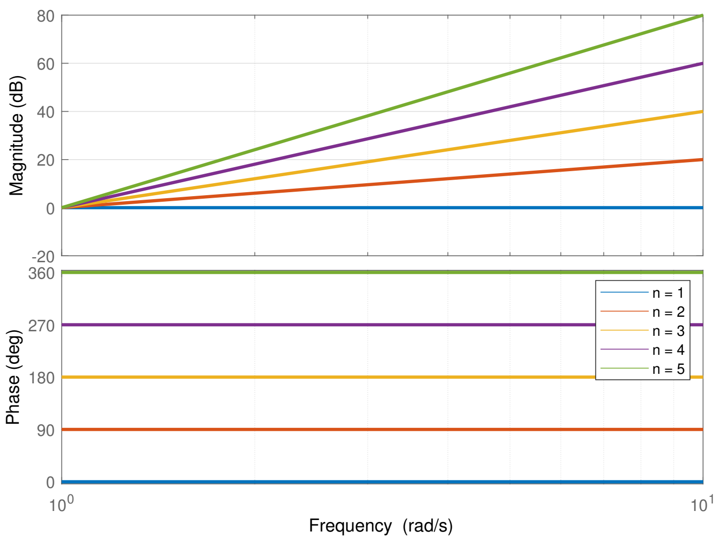


Fig. 17: *Bode plot of* $G(s) = s^n$.

6.3 Bode Plots for Integrator, $G(s) = (\frac{1}{s})^n$

- **Gain plot**:

$$20\log_{10}|G(j\omega)| = 20\log_{10}|(\frac{1}{j\omega})^n| = -n \cdot 20\log_{10}|(j\omega)| = \boxed{-n \cdot 20\log_{10}\omega}$$

- **Phase plot**: $\angle G(j\omega) = \angle(\frac{1}{j\omega})^n = \boxed{-n \cdot 90°}$
- Gain difference Δ over the interval $[\omega_1, \omega_2]$:

$$\Delta = -n \cdot 20(\log_{10}\omega_1 - \log_{10}\omega_2) = \boxed{-n \cdot 20\log_{10}\frac{\omega_1}{\omega_2}}$$

This gives the slope$= -n \cdot 20$ [dB/decade]

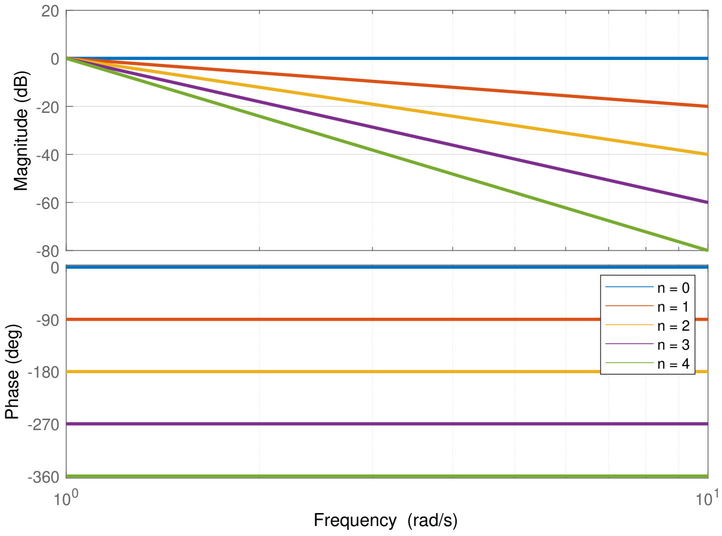


Fig. 18: *Bode plot of* $G(s) = (\frac{1}{s})^n$.

6.4 Bode Plots for 1st Order System, $G(s) = \frac{K}{Ts+1}$

- **Gain plot**:

$$20\log_{10}|G(j\omega)| = 20\log_{10}|\frac{K}{Ts+1}| = 20\log K - 20\log\sqrt{T^2\omega^2+1}$$
$$\approx \begin{cases} 20\log K & T\omega << 1 \\ 20\log K - 20\log T\omega & T\omega >> 0 \end{cases}$$

- **Phase plot**:

$$\angle G(j\omega)\angle\frac{1}{Tj\omega+1} = \frac{-T\omega j+1}{T^2\omega^2+1} = \tan^{-1}(-T\omega) \approx \begin{cases} 0^\circ & T\omega << 1 \\ -45^\circ & \omega = \omega_c = \frac{1}{T} \\ -90^\circ & T\omega >> 0 \end{cases}$$

- Gain difference Δ over the interval $[\omega_1, \omega_2]$:

$$\Delta = -20(\log_{10}\omega_1 - \log_{10}\omega_2) = \boxed{-20\log_{10}\frac{\omega_1}{\omega_2}}$$

This gives the slope$= -20$ [dB/decade]

- Corner frequency: $20\log T\omega_c = 0 \rightarrow \omega_c = \frac{1}{T}$

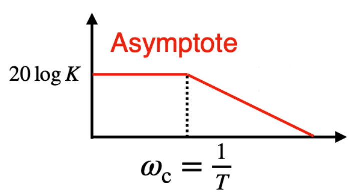


Fig. 19: *Corner frequency,* ω_c

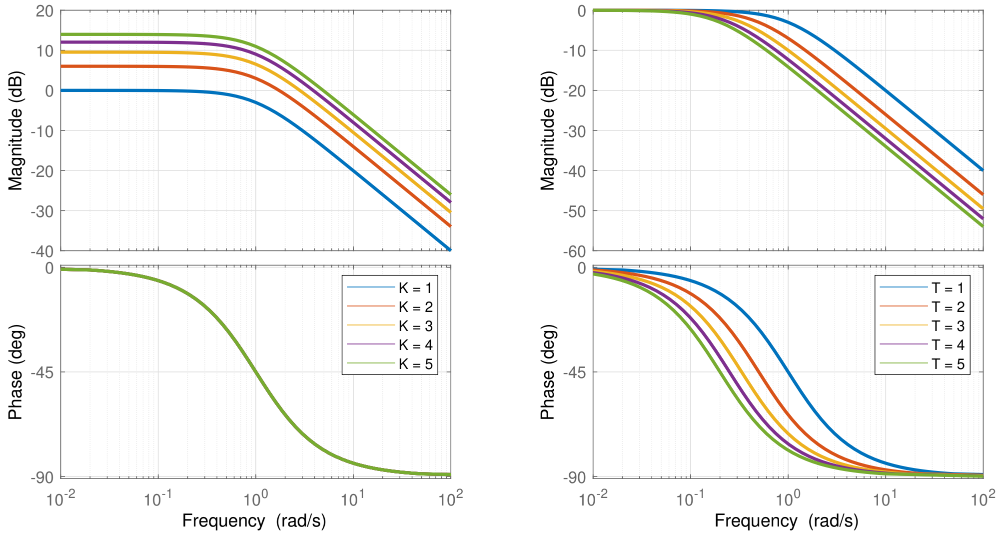


Fig. 20: *Bode plot for* $G(s) = \frac{K}{Ts+1}$. ***Left:*** $T = 1$, $K \in [1, 5]$. ***Right:*** $K = 1$, $T \in [1, 5]$.

6.5 Bode Plots for 1st Order Factor, $G(s) = (Ts + 1)^n$

- **Gain plot**:

$$20 \log_{10}|G(j\omega)| = n{\cdot}20 \log_{10}|Tj\omega{+}1| = n{\cdot}20 \log \sqrt{T^2\omega^2 + 1} \approx \begin{cases} 0 & T\omega << 1 \\ n \cdot 20 \log T\omega & T\omega >> 0 \end{cases}$$

- **Phase plot**:

$$\angle G(j\omega) = \angle(Tj\omega + 1)^n = n \cdot \tan^{-1}(T\omega) \approx \begin{cases} 0° & T\omega << 1 \\ n \cdot 45° & \omega = \omega_c = \frac{1}{T} \\ n \cdot 90° & T\omega >> 0 \end{cases}$$

- Corner frequency $\omega_c = \frac{1}{T}$, slope = $n \cdot 20$ [dB/decade]

6.6 Bode Plots for 2nd Order System, $G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Write the transfer function as a product of basic factors:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{-r^2 + 2\zeta jr + 1} = \frac{K}{(1 - r^2) + 2j\zeta r}$$

where r is known as the normalized frequency: $r = \frac{\omega}{\omega_c}$.

- **Bode plot**:

$$20 \log_{10}|G(j\omega)| = 20 \log_{10} \left| \frac{K}{(1 - r^2) + 2j\zeta r} \right| \approx \begin{cases} 20 \log K & r << 1 \\ 20 \log K - 40 \log r & r >> 0 \end{cases}$$

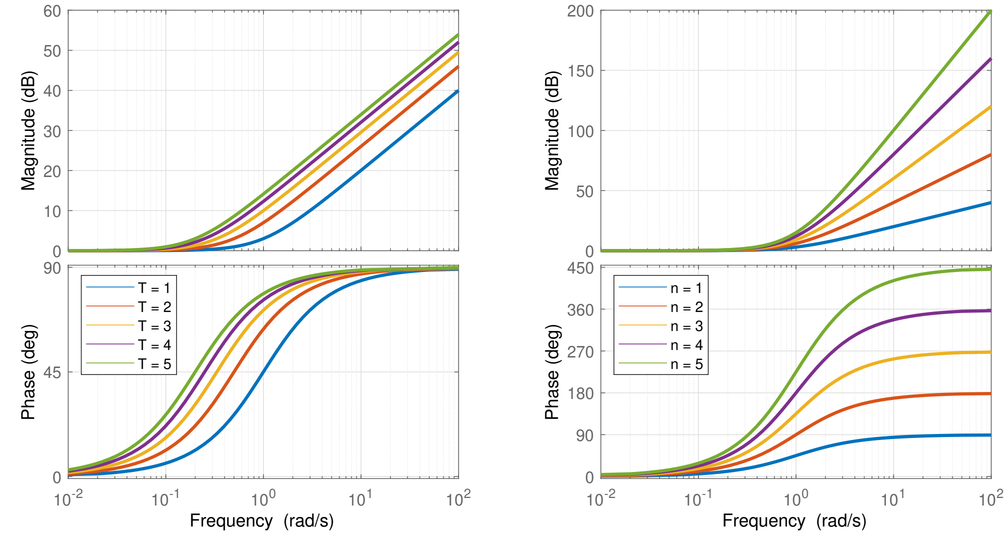


Fig. 21: *Bode plot for* $G(s) = (Ts+1)^n$. ***Left:*** $n = 1$**,** $T \in [1, 5]$**.** ***Right:*** $T = 1$**,** $n \in [1, 5]$**.**

- **Phase plot**:

$$\angle G(j\omega) = \angle \frac{K}{(1-r^2)+2j\zeta r} = \tan^{-1}(\frac{-2r\zeta}{1-r^2}) \approx \begin{cases} 0^\circ & r << 1 \\ -90^\circ & r = r_c = 1 \\ -180^\circ & r >> 0 \end{cases}$$

- Resonant frequency ω_r at which the magnitude becomes max:

$$\frac{d|G(j\omega)|}{dr} = 0 \rightarrow r^2 = q - 2\zeta^2$$

$$\omega_r = \omega_n \sqrt{1-2\zeta^2}$$

Resonant peak magnitude at $|G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

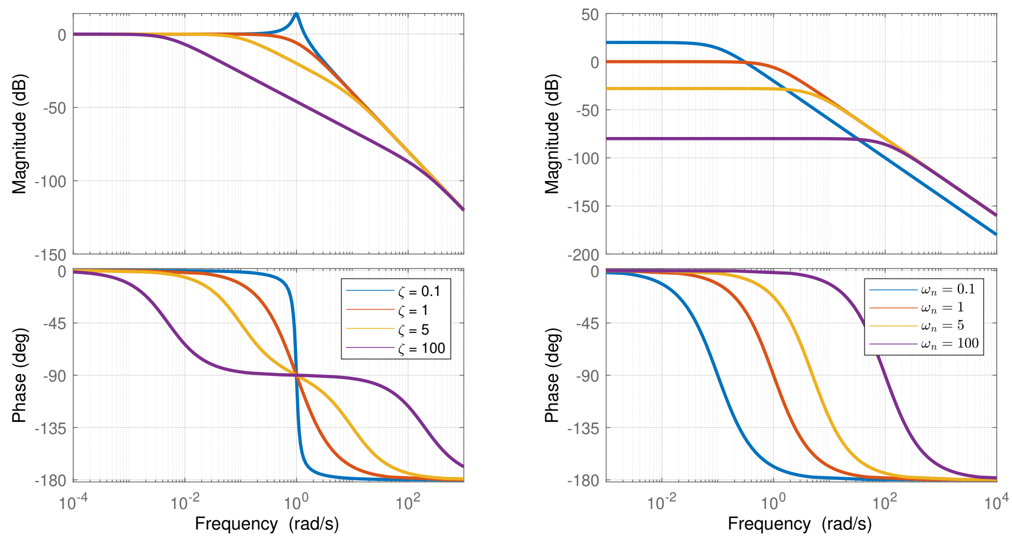


Fig. 22: *Bode plot for* $G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. ***Left:*** $\omega_n = 1$, ζ *varies.* ***Right:*** $\zeta = 1$, ω_n *varies.*

6.7 Drawing Approximate Bode Plots

1. Write the transfer function as a product of basic factors;

Example

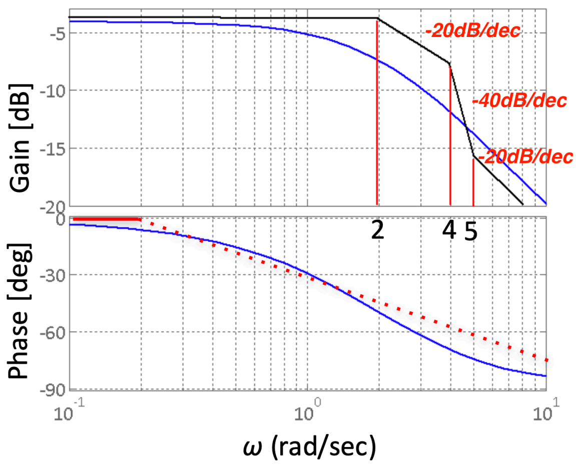
$$G(s) = \frac{s+5}{(s+2)(s+4)} = \frac{5(0.2s+1)}{2(0.5s+1).4(0.25s+1)} = \frac{0.625.(0.2s+1)}{(0.5s+1)(0.25s+1)}$$

2. Identify the corner frequency for each factor;

Example cont'd

$$\omega_c = \frac{1}{0.5}, \frac{1}{0.25}, \frac{1}{0.2} = 2, 4, 5$$

3. Draw the **asymptotes between the corner frequencies** and **add the individual plots**.



7 Stability Margins

When designing a control system, the first thing we want to ensure is the **stability** of the closed-loop system.

- Stability margins measure how close the system is to instability: less margin, less stable
- There are 2 ways to make the system unstable:
  - **Increase controller gain**, this reduces the **gain margin**.
  - **Increase time delay**, this reduces the **phase margin**.

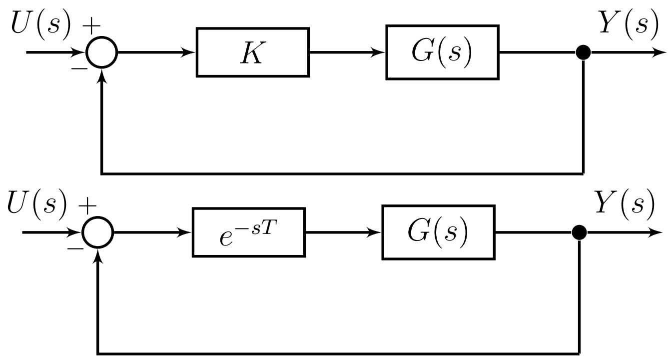
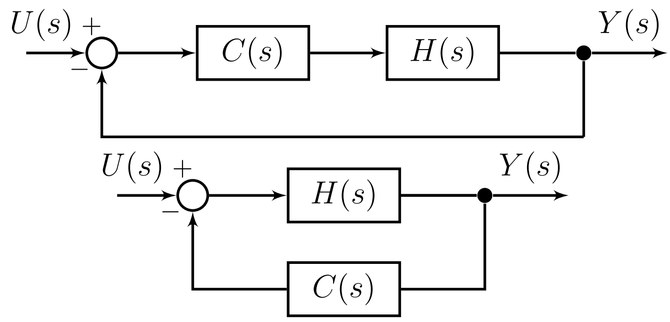


Fig. 23: *Two ways to make the system unstable*

- Two closed-loop, feedback systems have same characteristic functions. The open-loop transfer function, $C(s)H(s)$, determines the stability of the closed-loop systems:



$$G(s) = \frac{C(s)H(s)}{1 + C(s)H(s)} \qquad G(s) = \frac{H(s)}{1 + C(s)H(s)}$$

The closed-loop system becomes unstable if the gain of the closed loop system is ∞. Specifically, in this case, denominator $\rightarrow 0$:

$$1 + C(s)H(s) = 0 \quad \leftrightarrow \quad \underbrace{C(s)H(s)}_{\text{open-loop transfer function}} = -1 \quad \leftrightarrow \quad \begin{cases} |C(s)H(s)| = 1 \\ \angle C(s)H(s) = -180^\circ \end{cases}$$

- To make the Bode plot of $G_D(s) = C(s)H(s)$:

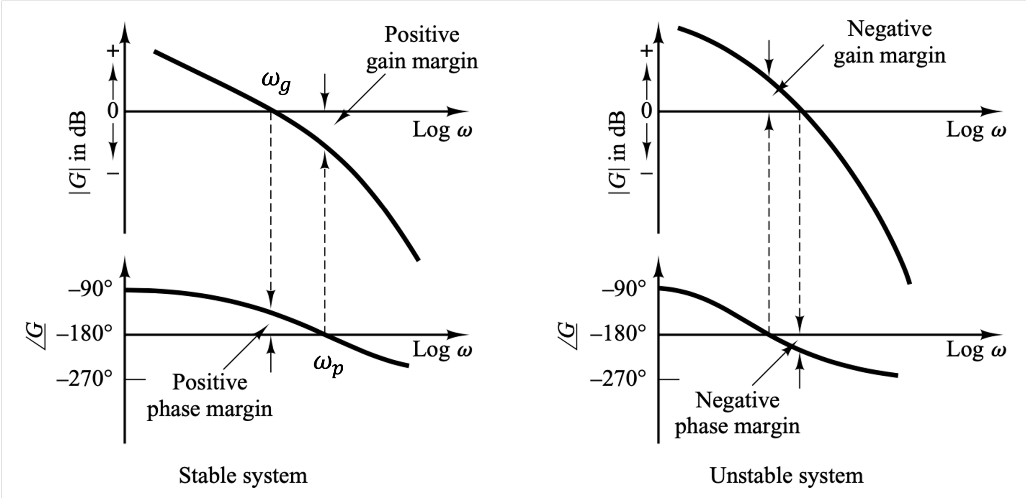


Fig. 24: *Stability margins for stable and unstable systems*

- Stability margin for stable systems: before losing the stability, we can
  1. add the same amount of phase delay as the **positive phase margin**, at ω, where $|G(\omega)| = 0$ dB
  2. add the same amount of gain as the **positive phase margin**, at ω, where $\angle G(\omega) = -180°$
- Stability margin for unstable systems: the system can become stable if we
  1. reduce the same amount of phase delay as the **negative phase margin**, at ω, where $|G(\omega)| = 0$ dB
  2. reduce the same amount of gain as the **negative gain margin**, at ω, where $\angle G(\omega) = -180°$

7.1 Gain margin

The gain margin is the gain relative to 0 dB when $\angle G(j\omega_p) = -180°$:

$$GM = -20 \log_{10}|G(j\omega_p)|$$

where ω_p is known as **phase cross-over frequency**, $\angle G(j\omega_p) = -180°$.

7.2 Phase margin

The **phase margin** is the phase relative to 180° when $|G(j\omega_g)| = 1$:

$$PM = \angle G(j\omega_g) - (-180°) = \angle G(j\omega_g) + 180°$$

where ω_g is known as the **gain cross-over frequency**, $20 \log|G(j\omega_g)| = 0$.

7.3 Delay margin

Delay margin is the translation of phase margin in the delay time.

$$T = \frac{PM}{\omega_g}\frac{\pi}{180°}$$

The system becomes unstable when the open-loop transfer function satisfies $e^{sT}G(s) = -1$ at the gain cross-over frequency, ω_g.

$$
\begin{aligned}
e^{-j\omega_g T}G(j\omega_g) = -1 \quad \longrightarrow \quad \angle(e^{j\omega_g T}G(j\omega_g)) &= \angle(e^{j\omega_g T}) + \angle G(j\omega_g) \\
&= -\omega_g T \cdot \frac{180^\circ}{\pi} + PM - 180^\circ \\
&= -180^\circ
\end{aligned}
$$

8 State-space Representation

- **State-space representation** is a convenient way to describe **multi-input/multi-output** LTI systems in time domain using matrix algebra.
- The system is described as a set of inputs, $\mathbf{u}(t)$, outputs, $\mathbf{y}(t)$, and state variables, $\mathbf{x}(t)$.
  - **State variables** is a set of variables that uniquely determines the current condition of the system (*e.g.* position, temperature, protein concentration)

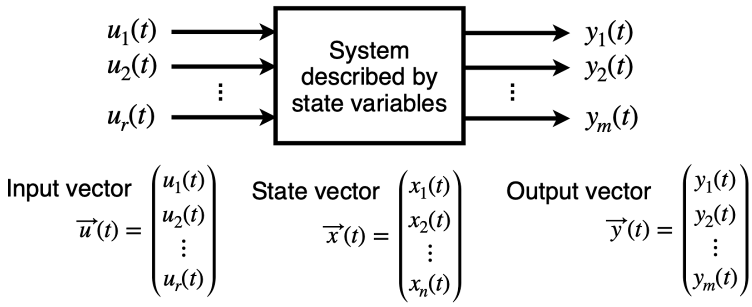


Fig. 25: *Multi-input/multi-output systems*

- The dynamics of state variables can be described by the 1st-order differential equations, known as the **state equation**.
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = A\mathbf{x} + B\mathbf{u}$$
  - A is a $n \times n$ matrix, known as the **state matrix**, $A \in \mathbb{R}^{n\times n}$.
  - B is a $n \times r$ matrix, known as the **input matrix**, $B \in \mathbb{R}^{n\times v}$.
- **Output equation**:
$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$
  - C is a $m \times n$ matrix, known as the **output matrix**, $C \in \mathbb{R}^{m\times n}$.
  - D is a $m \times r$ matrix, known as the **feedthrough matrix**, $D \in \mathbb{R}^{m\times r}$.
- **The transfer function** can be derived through taking the Laplace transform to the state-space model:
$$G(s) = \frac{\mathbf{y}(s)}{\mathbf{u}(s)} = C(sI - A)^{-1}B + D$$

Example 8.1

$$G(s) = \frac{s+4}{s^2+3s+2} \quad \longleftrightarrow \quad (s^2+3s+2)Y(s) = (s+4)U(s)$$

This gives us a 2nd order O.D.E.

$$\ddot{y} + 3\dot{y} + 2y = \dot{u} + 4u$$

There are an infinite number of possible state-space models.

8.1 Canonical Forms

Given a transfer function, there are infinite number of possible state-space representations.

- A system is **controllable** if there exists an input that can move its state variable from an initial state to any arbitrary state in a finite time.
- A system is **observable** if its states at any time point can be determined by observing the output over a finite interval of time.

For the system:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + ... + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n}$$

- The controllable canonical form is given by:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_n & b_{n-1} & b_{n-1} & \dots & b_1 \end{pmatrix} \mathbf{x}$$

Example 8.2

The system

$$G(s) = \frac{s+4}{s^2+3s+2}$$

can be written as (in controllable canonical form)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0u$$

To verify this, start from the definition of the transfer function:

$$\begin{aligned} G(s) &= \frac{\mathbf{y}(s)}{\mathbf{u}(s)} = C(sI - A)^{-1}B + D \\ &= \begin{pmatrix} 4 & 1 \end{pmatrix} \{sI - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \end{pmatrix} \frac{1}{s(s+3)+2} \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{s+4}{s(s+3)+2} \end{aligned}$$

- The observable canonical form is given by:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \ddots & \vdots & -a_{n-1} \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -a_2 \\ 0 & \dots & 0 & 1 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix} \mathbf{x}$$

Example 8.3

The system

$$G(s) = \frac{s+4}{s^2+3s+2}$$

can be written as (in observable canonical form)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0u$$

To verify this, start from the definition of the transfer function:

$$\begin{aligned} G(s) = \frac{\mathbf{y}(s)}{\mathbf{u}(s)} &= C(sI-A)^{-1}B + D \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \{sI - \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}\}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s & 2 \\ -1 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{1}{s(s+3)+2} \begin{pmatrix} s+3 & -2 \\ 1 & s \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \frac{s+4}{s(s+3)+2} \end{aligned}$$

8.2 Stability of State-Space Model

The system is stable if all eigenvalues of **A** satisfies $\Re(\lambda_i) < 0$. *i.e.* All eigenvalues of A are on the left-side of the s-plane.

There are 2 ways to find the matrix exponential:

- Diagonalisation of **A**;
- Using the identity: $e^{At} = \mathcal{L}^{-1}(sI-A)^{-1}$.

8.2.1 Solutions of State-space Model

$$\mathbf{x}(t) = \underbrace{e^{At}\mathbf{x}(0)}_{\text{zero-input response}} + \underbrace{\int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\mathrm{d}\tau}_{\text{zero-state response}}$$

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}(0) + C\int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\mathrm{d}\tau + D\mathbf{u}(\tau)$$

Example 8.4

Find the response of the system where

$$\mathbf{x} = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

and $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $u(t) = 5$, $y = \begin{pmatrix} 2 & 1 \end{pmatrix}\mathbf{x}$

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\mathrm{d}\tau$$

Since[a]

$$e^{At} = \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{pmatrix}$$

Therefore

$$\mathbf{x}(t) = \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-2\tau} & 0 \\ e^{-\tau} - e^{2\tau} & e^{-\tau} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} \mathrm{d}\tau$$

$$= \begin{pmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{pmatrix} \begin{pmatrix} \int_0^t 5e^{2\tau}\mathrm{d}\tau \\ \int_0^t (5e^{\tau} - 5e^{2\tau})\mathrm{d}\tau \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{2}(1 - e^{-2t}) \\ \frac{5}{2}(1 + e^{-2t}) - 5e^{-t} \end{pmatrix}$$

$$y = (2\ 1)\mathbf{x} = (2\ 1) \begin{pmatrix} \frac{5}{2}(1 - e^{-2t}) \\ \frac{5}{2}(1 + e^{-2t}) - 5e^{-t} \end{pmatrix} = \frac{15}{2} - \frac{5}{2}e^{-2t} - 5e^{-t}$$

Matrix exponential: $e^{tA} = Ve^{tD}V^{-1}$

$$tA = V(tD)V^{-1}$$

When eigenvalues of A are λ_1, λ_2, ...Eigenvalues of tA are $t\lambda_1$, $t\lambda_2$, ...

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{V(tA)^k V^{-1}}{k!} = V\left(\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\right)V^{-1}$$

$$\sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} (t\lambda_1)^k & 0 & \dots & 0 \\ 0 & (t\lambda_2)^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (t\lambda_n)^k \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(t\lambda_1)^k}{k!} & 0 & \dots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(t\lambda_2)^k}{k!} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{k=0}^{\infty} \frac{(t\lambda_n)^k}{k!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{t\lambda_n} \end{pmatrix}$$

$$= e^{tD}$$

[a] derivation see below

8.2.2 Impulse Response

Transfer function

$$G(s) = \frac{\mathbf{y}(s)}{\mathbf{u}(s)} = C(sI - A)^{-1}B + D$$

Impulse response

$$\mathbf{y}(t) = Ce^{At}B + D$$

$$\frac{d}{\mathrm{d}t}e^{At} = Ae^{At} \xrightarrow{\mathcal{L}} s\mathcal{L}(e^{At}) - e^{A0} = A\mathcal{L}(e^{At})$$

$$(sI - A)\mathcal{L}(e^{At}) = e^{A0} = I$$

$$\mathcal{L}(e^{At}) = (sI - A)^{-1}$$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

8.2.3 Stability of State-space Models

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ &= C\frac{1}{det(sI - A)}(sI - A)^T_{cofactor}B + D \end{aligned}$$

Poles s of $G(s)$ = Eigenvalues λ of A.

$$det(sI - A) = det(\lambda I - A) = 0$$

System is stable *if and only if* $\Re(\lambda_i) < 0$ for all eigenvalues λ_i of A.

8.2.4 Pole Placement

- **SISO system**: Poles of the closed-loop system as a function of the gain K: $\frac{Y(s)}{R(s)} = \frac{G(s)}{1+KG(s)}$.

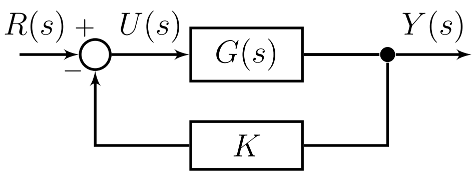


Fig. 26: *SISO system*

- **MIMO system**:

$$\dot{x} = Ax + Bu = Ax + B(r - Kx) = (A - BK)x + Br$$

Eigenvalues of the matrix $(A - BK)$ determines the closed-loop stability.

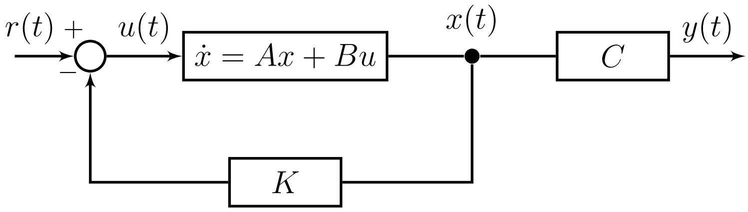


Fig. 27: *MIMO system*

Example 8.5

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -0.5 & -0.8 \\ 0.8 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Poles at $-0.25 \pm 0.76j$.
Design a state-feedback:

$$\hat{u} = \begin{pmatrix} k_1 & k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So that the closed-loop system has the poles at -1 and -2:

$$A - BK = \begin{pmatrix} -0.5 & -0.8 \\ 0.8 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} k_1 & k_2 \end{pmatrix} = \begin{pmatrix} -0.5 - k_1 & -0.8 - k_2 \\ 0.8 & 0 \end{pmatrix}$$

$$\det\{\lambda I - (A - BK)\} = \begin{vmatrix} \lambda + 0.5 + k_1 & 0.8 + k_2 \\ -0.8 & \lambda \end{vmatrix} = (\lambda + 1)(\lambda + 2)$$

$$\lambda(\lambda + 0.5 + k_1) + 0.8(0.8 + k_2) = (\lambda + 1)(\lambda + 2)$$

This gives two solutions: $k_1 = 2.5$ and $k_2 = 1.7$

8.3 Decouple Interconnected Systems

8.3.1 Relationship Between Equivalent Systems

The system

$$G(s) = \frac{1}{(s-1)(s-2)}$$

can be written in controllable canonical form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}}_{A_1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and observable canonical form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}}_{A_2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Since the two forms come from one transfer function, the system stability must be the same. More importantly, A matrix determine the stability of the system.

Relationship between A_1 and A_2:

$$\begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -5 & 1 \\ 1 & 1 \end{pmatrix}$$

$$TA_1 = A_2T \quad \leftrightarrow \quad A_1 = T^{-1}A_2T$$

Moreover, both matrices A_1, A_2 has same eigenvalues at 1 and 2. So A_1, A_2 can be digonalized to the same matrix.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

with the relation

$$A_1V_1 = V_1D \qquad A_2V_2 = V_2D$$

8.3.2 Decoupling of Interconnected Systems (*from Math 2*)

Decoupling of interconnected systems are done by matrix digonalization.

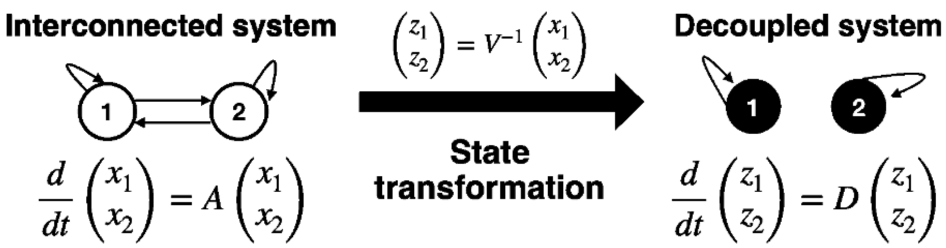


Fig. 28: *De-couple an interconnected system*

Decoupling of the interconnected system allows us to analyze the dynamics of a large system by only considering smaller systems.

Example 8.6

- Interconnected system:

$$\frac{d}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{A_1=\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_1 + 3x_2 \end{pmatrix}$$

$\dot{x}_1$ and $\dot{x}_2$ depend on x_1 and x_2, this means the system is interconnected.

- Decoupled system:

$$\frac{d}{\mathrm{d}t}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = D\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \xrightarrow{D=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ 2z_2 \end{pmatrix}$$

$\dot{z}_1$ only depends on z_1 and $\dot{z}_2$ only depends on z_2, this means the system is decoupled.

- The matrix D can be obtained by digonalization of matrix A

$$D = V^{-1}AV$$

where V is the eigenvector.

- The relationship between z_1, z_2 and x_1, x_2 satisfies:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = V^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In this case:

$$V = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

8.4 Controllability and Observabiliy

8.4.1 Controllability

- A system is controllable if there exists an input that can move its state variable from an initial state to any arbitrary state in a finite time;
- A systems is controllable if the controllability matrix is of full rank

$$rank(B\ AB\ A^2B\ ...\ A^{n-1}B) = n$$

8.4.2 Observability

- A system is observable if its states at any time point can be determined by observing the output over a finite interval of time.
- A system is observable if the observability matrix is of full rank.

$$\text{rank}\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$$

8.4.3 Stabilisability and Detectability

- A system is stabilisable if all the uncontrollable states are stable.
- A systems is detectable if all the unobservable states are stable.