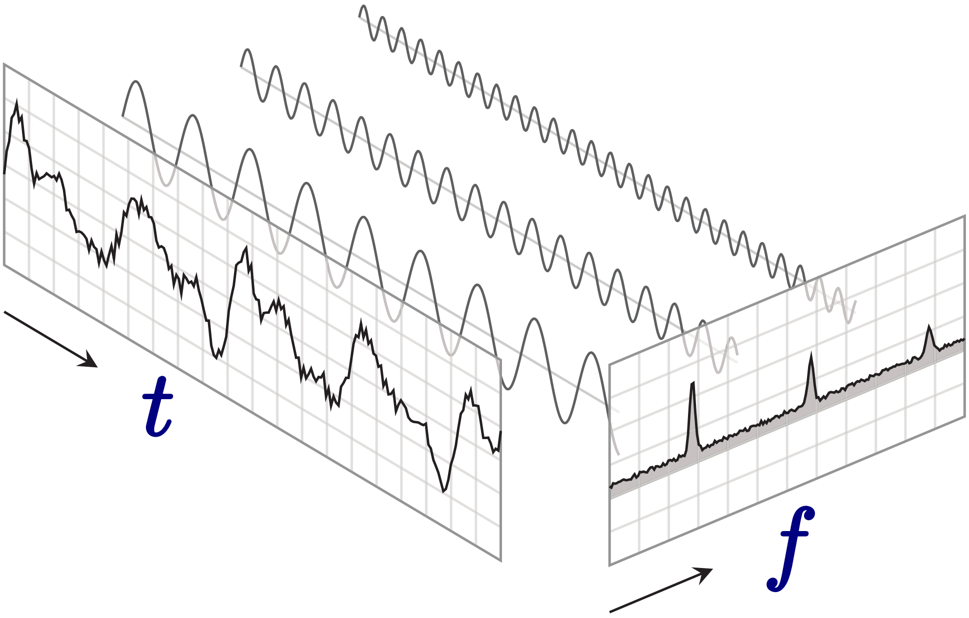
Imperial College London

Signals and Control - Signals



Binghuan Li

binghuan.li19@imperial.ac.uk

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Preface, Rationale, & Acknowledgements

This document was initially constructed and released in November 2020 as a collection of my class notes for the first element *Signals and Systems* in the module *BIOE50011-Signals and Control* (same syllabus applied to the module *BIOE50005 - Mathematics and Engineering 2*) at the Department of Bioengineering, Imperial College London. Multiple insightful suggestions and requirements came in the subsequent years regarding the update and maintenance of this document, and surprisingly, I am still able to pick up those contents and revisit them as if they are from the lectures I attended yesterday.

For anyone who is using this document - I really wish you can go well with the signals, rather than just learn for the somehow tricky progress test or the final exam, but really appreciate the magic and use it to facilitate your own research. Believe it or not, you are not likely to regret not learning signals well before permanently quitting them and moving to cardiovascular fluid research (Yea, that's my research).

I would like to express my deepest appreciation to *Prof. Dario Farina* for his help in reviewing these notes with the highest profession. I would also like to extend my sincere thanks to *Rea Tresa*, *Ben Ford*, and *Elisa Soliani* for their comments. Their work significantly optimized the structure. Finally, my special thanks go to *Haroon Chughtai*. His notes greatly enlightened me when preparing this new set.

A few typos and grammar issues have been fixed in Oct. 2023, and subsection 3.6 has been extensively reviewed with fig.12 reproduced, and equations re-typesetted. I have personally been confused by the definition of cross-correlation and auto-correlation for years until I found an intuitive breakdown from *Prof. A. A. Bharath*'s old slides.

The LaTeX files are now accessible on my GitHub repository. I hope it helps. Please report typos and inconsistencies to binghuan.li19@imperial.ac.uk.

March & October, 2023

Cover image: The Fourier Analysis, modified and reproduced with the original TikZ source code provided by *Tobit Flatscher* from Tex-StackExchange.

Contents

1 Definition, Classification, and Properties of Signals

1.1 Definition of signals

- Signals describe physical phenomena as patterns of variations of some form.
- Mathematically, signals are **functions** of one or more independent variables.
- For example, a signal $s(t)$ can be a function of the continuous independent variable time $t \in [\alpha, \beta]$ A two-dimensional signal $f(x, y)$ can be a function of two spatial coordinates x, y.

1.2 Continuous and Discrete-time Signals

- Signals can be a function of the **continuous time** variable, in which case we will use the notation $x(t)$ with $t \in \mathbb{R}$; or of the **discrete time** variable, in which case we will use the notation $x[n]$ with $n \in \mathbb{Z}$. (Figure 1)

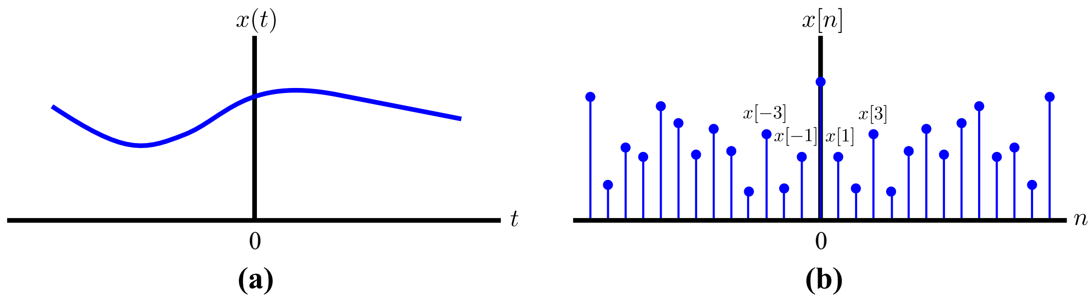


Fig. 1: *Illustration of a (a) continuous-time signal $x(t)$ and (b) a discrete-time signal $x[n]$*

- Discrete-time signals are often (but not necessarily) a sampling of continuous-time signals.

$$x[n] = x_c(nT), \quad -\infty < n < +\infty$$

  T is sampling period.

- A discrete-time signal can be represented as a sequence of numbers, or, a vector.

1.2.1 Digital Signals

- When we discuss a digital signal, we often mean the signal that has been **sampled** (captured at regular points in time) and **quantisised**.
- When one refers to a 12-bit signal, they are referring to the number of amplitude quantisation levels.
- Sampling a continuous signal may be done **without losing any information** from the original signal. Conversely, quantisation always implies **losing information**.

We focus on the signals of one independent variable!

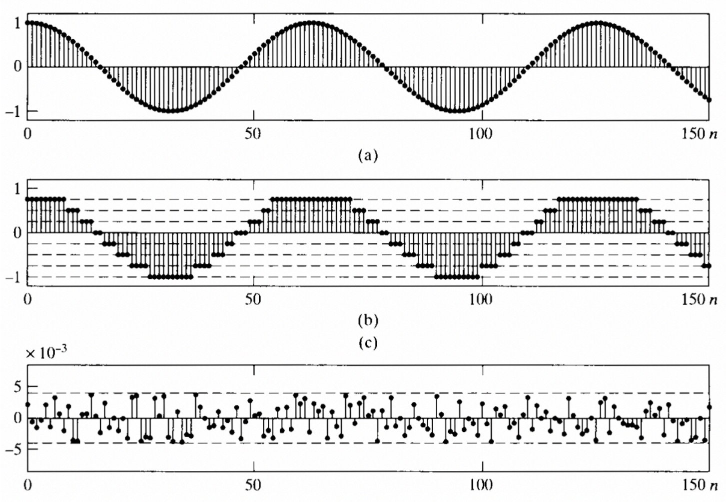


Fig. 2: *Sampling and quantisation*

1.3 Deterministic and Stochastic Signals

- **Deterministic**: a signal that **can be predicted** exactly (an analytical formulation exists).
  - Example: $x(t) = \sin(2\pi t)$
- **Stochastic**: a signal that **cannot be predicted** exactly before it has "occurred"; any signal that conveys information to us when we observe it.
  - Example: Thermal noise across a resistor, EEG traces, *etc.*
- We can often meaningfully describe the statistical properties of stochastic signals by building a model of their generation (stochastic processes).

We will mainly deal with deterministic signals in this course!

1.4 Periodic Signals

- A periodic continuous-time signal $x(t)$ has the property that there is a positive value of T for which $x(t) = x(t+T)$ for all values of t (similar definition for discrete-time signals).
- A periodic signal has the property that it is unchanged by a time shift of T, we will say that $x(t)$ is periodic with period T. (Figure 3)

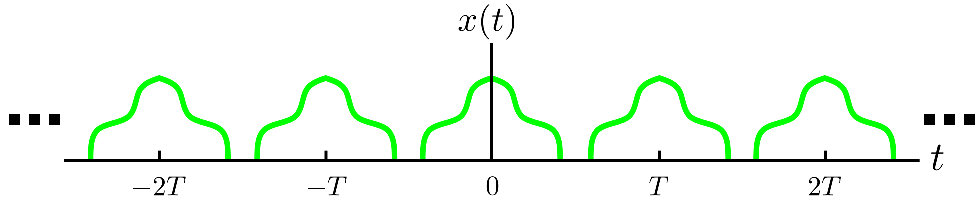


Fig. 3: *A periodic signal with the period T*

1.5 Signal Energy and Power

For a continuous-time signal $x(t)$ for $t_1 \leq t \leq t_2$ and for a discrete-time signal $x[n]$ for $n_1 \leq n \leq n_2$, energy and power can be represented as follows:

$$\text{Energy(continuous time)} = \int_{t_1}^{t_2} |x(t)|^2 \, \mathrm{d}t$$

$$\text{Power(continuous time)} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 \, \mathrm{d}t$$

$$\text{Energy(discrete time)} = \sum_{n=n_1}^{n_2} |x[n]|^2$$

$$\text{Power(discrete time)} = \frac{1}{n_2 - n_1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

Electrical circuit analogy

We get the conclusion above from the calculation for electrical power and energy. Let $v(t)$ and $i(t)$ represent the voltage and current across the resistor of resistance R.

- The instantaneous power across the resistor is the product $v(t)i(t)$, which is proportional to $v^2(t)$.
- The total energy

$$\int_{t_1}^{t_2} \frac{1}{R} v^2(t) \, \mathrm{d}t$$

- The average power

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R} v^2(t) \, \mathrm{d}t$$

Similar properties can be applied to any continuous-time signals and discrete-time signals.

- Often, the signals are directly related to physical quantities capturing power and energy in a physical form.
- These properties are important characteristics of signals, even if in some cases do not reflect physical energy or power.

1.5.1 Energy and Power of a Generic Signal

Extend the range to: $-\infty < t < +\infty$ or $-\infty < n < +\infty$

- In continuous time:

$$\text{Energy} = \int_{-\infty}^{+\infty} |x(t)|^2 \, \mathrm{d}t$$

$$\text{Power} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

- In discrete time:

$$\text{Energy} = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

$$\text{Power} = \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

We will use the mathematical definitions above, regardless of the direct physical meaning of each term!

2 Types of Signals

2.1 Periodic Complex Exponential Signals in Continuous-time Domain

$$x(t) = e^{j\omega_0 t}$$

- Periodic, period $T = \frac{2\pi}{|\omega_0|}$.
- The signal $x(t) = e^{-j\omega_o t}$ has the same period.
- The complex exponential defined above is closely related to the sinusoidal signal:

$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

  which has the same period $T = \frac{2\pi}{|\omega_0|}$.

- The complex exponentials and sinusoidal signals have **infinite energy** and **finite power**.
  - Example: for the signal $x(t) = A\cos(2\pi\omega_0 t + \phi)$ with the period T_1,

$$\text{Power} = \frac{A^2}{2}$$

2.2 Periodic Complex Exponential Signals in Discrete-time Domain

$$x[n] = e^{j\omega_0 n}$$

And the sinusoidal signal becomes

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

- $n \in \mathbb{Z}$ (*i.e.,* n is an integer). Thus, $x[n]$ is the same signal for $\omega_0 + 2\pi k$ with $k \in \mathbb{Z}$. The frequency of oscillation in discrete time exponentials does not increase monotonically but is limited to 2π.
- $x[n]$ is not always periodic.

2.3 The Unit Impulse in Discrete-time Domain

The **unit impulse function** (or, delta function) defined in the discrete-time domain (Figure 4) is

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

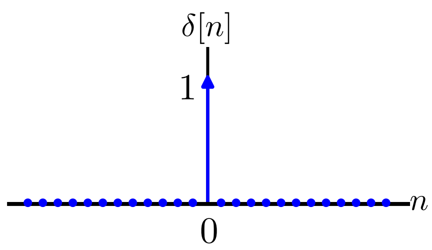


Fig. 4: *The unit impulse*

The discrete-time unit impulse **delayed by an integer** k (Figure 5) is defined as:

$$\delta[n-k] = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$$

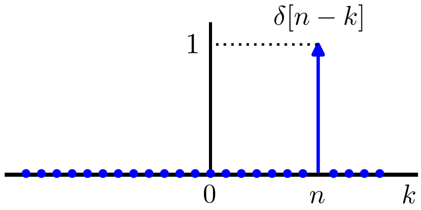


Fig. 5: *The unit impulse delayed by an integer k*

- For any discrete-time signal $x[n]$, we have

$$x[n]\,\delta[n-k] = x[k]\,\delta[n-k]$$

  This implies any signal multiplied by the unit impulse is zeroed for all time samples, apart from the integer time where the unit impulse is centred.

- From the property above, we have:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]\delta[n-k] &= \sum_{n=-\infty}^{\infty} x[k]\,\delta[n-k] \\ &= x[k] \underbrace{\sum_{n=\infty}^{\infty} \delta[n-k]}_{1,\ \text{when } k = n} \\ &= x[k] \end{aligned}$$

- Any arbitrary discrete-time signal can be expressed as the sum of **scaled** and **delayed** impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

2.4 The Unit Step in Discrete-time Domain

The **unit step function** defined in the discrete-time domain (Figure 6) is

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = \sum_{k=0}^{+\infty} \delta[n-k]$$

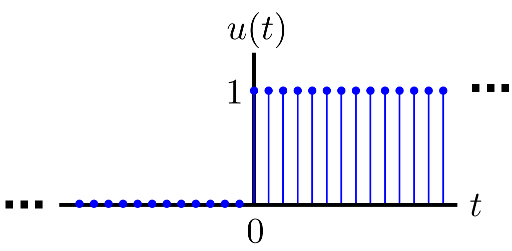


Fig. 6: *Unit step in discrete-time domain*

- The discrete-time unit impulse function is the *derivative* of the discrete-time unit step function.

$$\delta[n] = u[n] - u[n-1]$$

2.5 The Unit Step in Continuous-time Domain

The **unit step function** defined in the continuous-time domain (Figure 7) is

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

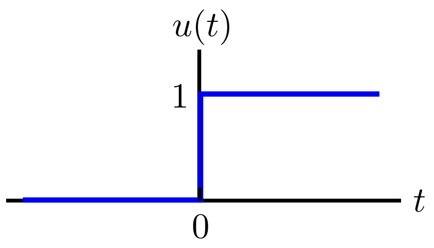


Fig. 7: *Unit step in continuous-time domain*

- The continuous-time unit step function has a discontinuity in $t = 0$, hence, it is **not differentiable**.
- The aforementioned issue could be addressed using the concept of *limit*. By expressing the delta function as the derivative of the unit step function over an infinitesimal period of time, Δ, we have

$$\delta_\Delta(t) = \frac{\mathrm{d}u_\Delta(t)}{\mathrm{d}t}$$

  where

  - The area of $\delta_\Delta(t)$ equals to 1 at any value of Δ.(Figure 8, right)
  - As $\Delta \to 0$, $u_\Delta(t) \to u(t)$. (gradient $\to 0$) (Figure 8, left)
  - As $\Delta \to 0$, the impulse $\delta_\Delta(t)$ becomes of shorter duration and higher amplitude:$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$

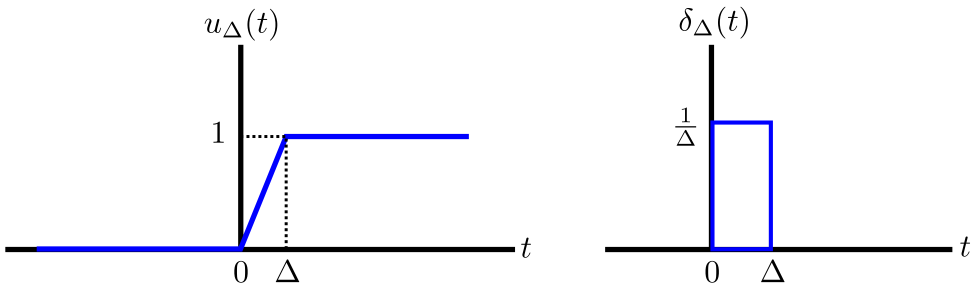


Fig. 8: *Continuous approximation to unit step*

- The unit step function can be reconstructed by integrating the delta function from negative infinity to t (Figure 9, left),

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

  where τ is a simply a *dummy* variable used to replace the notation of t.

- for the discrete-time case, the unit impulse in the continuous-time domain can be shifted along the time axis. The unit impulse shifted by the time delay is $\delta(t - \sigma)$.

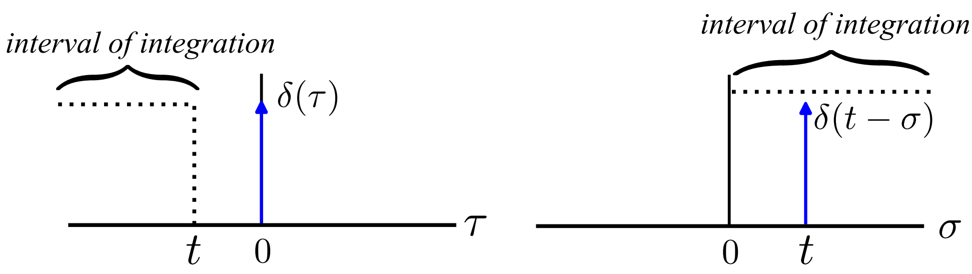


Fig. 9: *Unit impulse shifted by the time delay*

- In continuous-time domain: (similar to discrete-time domain)

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

As $\delta_\Delta(t) \rightarrow \delta(t)$: better approximation

$$x(t)\delta(t) = x(0)\delta(t)$$

More generally: with time-shifting

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

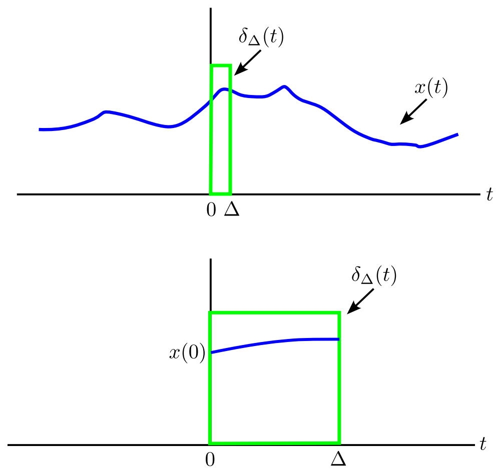


Fig. 10: *Continuous approximation of the unit impulse*

- We also obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\delta(t-t_0)\mathrm{d}t &= \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)\mathrm{d}t \\ &= x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0) \\ &= x(t_0)\end{aligned}$$

- This implies that any arbitrary continuous-time signal $x(t)$ can be represent as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau$$

2.5.1 Convolution

We define the following transformation between two signals (convolution):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\mathrm{d}\tau = x(t) * h(t)$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k] = x[n] * h[n]$$

Convolution in the time main is equivalent to the multiplication in the frequency domain!

For any continuous-time signal and any discrete-time signal:

$$x(t) = x(t) * \delta(t)$$

$$x[n] = x[n] * \delta[n]$$

by extension for arbitrary delays:

$$x(t - t_0) = x(t) * \delta(t - t_0)$$

$$x[n - k] = x[n] * \delta[n - k]$$

Summary of properties of the unit impulse

- For **multiplication**:

$$x(0) \cdot \delta(t) = x(t) \cdot \delta(t)$$

$$x[0] \cdot \delta[n] = x[n] \cdot \delta[n]$$

$$x(t_0) \cdot \delta(t - t_0) = x(t) \cdot \delta(t - t_0)$$

$$x[k] \cdot \delta[n - k] = x[n] \cdot \delta[n - k]$$

- For **convolution**:

$$x(t) = x(t) * \delta(t)$$

$$x[n] = x[n] * \delta[n]$$

$$x(t - t_0) = x(t) * \delta(t - t_0)$$

$$x[n - k] = x[n] * \delta[n - k]$$

3 Simple Operations on Signals

3.1 Transformations of the Time Variable

- Time delay
- Time reversal
- Time scaling

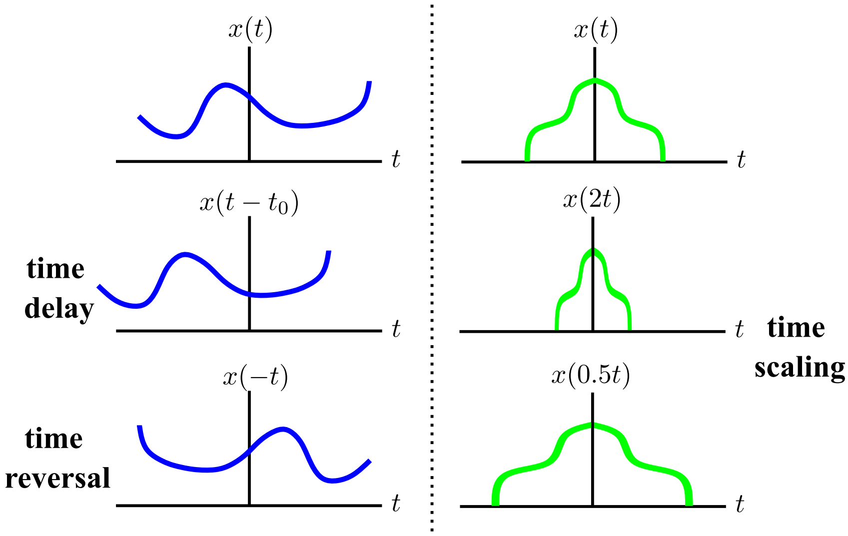


Fig. 11: *Transformations applies to the time variable: time delay (left middle), time reversal (left bottom), time scaling (right)*

3.2 Amplitude Transformation

$$y(t) = Ax(t) + B$$

where A, B are constants.

3.3 Linear Combination

- In continuous-time domain:

$$y(t) = a_1 x_1(t) + a_2 x_2(t) + ... + a_N x_N(t)$$

- In discrete-time domain:

$$y[n] = a_1 x_1[n] + a_2 x_2[n] + ... + a_N x_N[n]$$

where $a_i \in \mathbb{C}$, $i = 1, 2, ..., N$ are real or complex numbers.

3.4 Multiplication

- In continuous-time domain:
$$y(t) = x_1(t) \cdot x_2(t)$$
- In discrete-time domain:
$$y[n] = x_1[n] \cdot x_2[n]$$

This implies **instantaneous multiplication** for each time instant or each discrete time sample.

3.5 Scalar Products and Norms

3.5.1 Scalar Product and Norm of Vectors

- The **scalar product** between two 3-D vectors $\vec{A}$ and $\vec{B}$: projection of $\vec{A}$ on $\vec{B}$.
$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z = |\vec{A}| \cdot |\vec{B}| \cos(\phi)$$
- For vectors, the **Euclidean norm**, or **norm-2**, is the length of vectors in Euclidean space:
$$||\vec{A}||_2 = \sqrt{\vec{A} \cdot \vec{A}} = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

3.5.2 Scalar Product and Norm of Discrete-time Signals

- **Scalar product** (or, inner product) between two discrete-time signals, $x_1[n]$ and $x_2[n]$:
$$\langle x_1[n], x_2[n] \rangle = \sum_{n=-\infty}^{\infty} x_1[n] x_2^*[n]$$
- **Norm-2** for a discrete-time signal $x[n]$:
$$||x[n]||_2 = \sqrt{\langle x[n], x[n] \rangle} = \sqrt{\sum_{n=-\infty}^{\infty} |x[n]|^2} = \left(\sum_{n=-\infty}^{\infty} |x[n]|^2 \right)^{\frac{1}{2}}$$
  norm-2 is the square root of the **energy** of the signal
- **Norm-p** for a discrete-time signal, $x[n]$:
$$||x[n]||_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty$$
  - For $p = 1$ (norm-1):
$$||x[n]||_1 = \sum_{n=-\infty}^{\infty} |x[n]|$$
  - For $p \to \infty$: infinity norm / maximum norm:
$$||x[n]||_\infty = \max_n |x[n]|$$
- Norms are measures of the signal "strength". Each norm is a different way of measuring signal strength. *E.g.,* norm-2 is associated with the energy.
- The space of signals with a finite norm-p is called L^p space.

3.5.3 Scalar Product and Norm for Continuous-time Signals

- **Scalar product** (or, inner product) between two continuous-time signals, $x_1(t)$ and $x_2(t)$:

$$\langle x_1(t), x_2(t)\rangle \;=\; \int_{-\infty}^{+\infty} x_1(t)x_2^*(t)\mathrm{d}t$$

  where $x_2^*(t)$ denotes the complex conjugate of $x_2(t)$ - this can be ignored if the signal only has the real part.

- **Norm-2** for the continuous-time signal, $x(t)$:

$$||x(t)||_2 = \sqrt{\langle x(t), x(t)\rangle} = \sqrt{\int_{-\infty}^{+\infty} |x(t)|^2\mathrm{d}t} = \left(\int_{-\infty}^{\infty} |x(t)|^2\right)^{\frac{1}{2}}$$

- **Norm-p** for the continuous-time signal, $x(t)$:

$$||x(t)||_p = \left(\int_{-\infty}^{+\infty} |x(t)|^p\mathrm{d}t\right)^{\frac{1}{p}} \quad ; \quad ||x(t)||_\infty \;=\; \max_n |x(t)|$$

Question 3.1

The signal $f_1(t)$ is defined as

$$f_1(t) = \begin{cases} |t|, & |t| \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

and the signal $f_2(t)$ is defined as

$$f_2(t) = -2\sin(2t)$$

The inner product between the signals $f_1(t)$ and $f_2(t)$ is:

(a) 3
(b) 1
(c) π
(d) 0

3.6 Characterising Similarity/Difference Between Signals

3.6.1 Measuring the Similarity

Normalizing the scalar product between two *2-dimensional* vectors ($\vec{A}\cdot\vec{B}$) to the lengths of the vectors ($|\vec{A}|$ and $|\vec{B}|$) gives out the information of direction between two vectors,

$$\vec{A}\cdot\vec{B} = |\vec{A}|\cdot|\vec{B}|\cos(\phi) \quad \Rightarrow \quad \cos(\phi) = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|} \in [-1, 1]$$

- if $\cos(\phi) = 1$, $\vec{A}$ and $\vec{B}$ are parallel;
- if $\cos(\phi) = -1$, $\vec{A}$ and $\vec{B}$ are anti-parallel;

- if $\cos(\phi) = 0$, $\vec{A}$ and $\vec{B}$ are orthogonal.

A similar idea applies to measuring how similar two signals are - we *normalise* the scalar (inner) product between two signals by norm-2 of the signals. This is called the **normalised scalar product** between two signals[1].

- For two *continuous-time* signals,

$$\cos(\phi) = \frac{\langle x_1(t), x_2(t)\rangle}{||x_1(t)||_2 \; ||x_2(t)||_2};$$

- for two *discrete-time* signals,

$$\cos(\phi) = \frac{\langle x_1[n], x_2[n]\rangle}{||x_1[n]||_2 \; ||x_2[n]||_2}.$$

3.6.2 Cross-correlation Function and Normalized Cross-correlation Function

- In practical conditions, signals are commonly corrupted by noise.

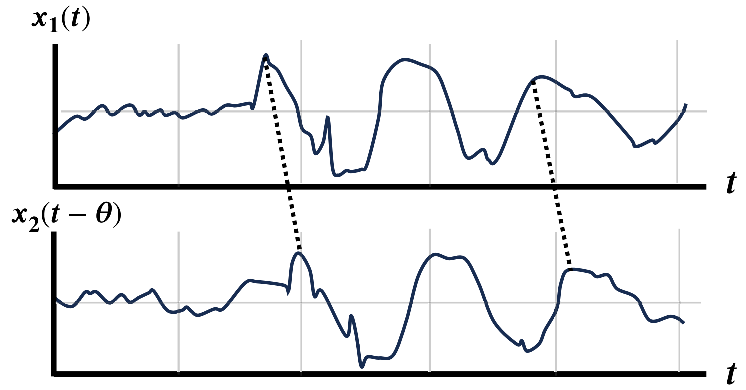


Fig. 12: *Two signals corrupted by noise, but they are correlated by shifting $x_2(t)$ with the time delay θ*

- A possible estimate of the delay between the two signals is the time interval by which we need **to shift one of the signals so that it is maximally similar to the other**, *i.e.*, determining the shift, θ, at which two signals are in best temporal alignment.

Cross-correlation function, $c(\theta)$ The cross-correlation function is defined as the scalar product between the signal $x_1(t)$ and the *shifted* signal $x_2(t-\theta)$,

$$c(\theta) = \langle x_1(t), x_2(t-\theta)\rangle,$$

where our objective is finding the best value of θ that could maximise the cross-correlation function, *i.e.*, find the max similarity between two signals,

$$\theta_{\text{best}} = \arg\max_{\theta\in\mathbb{R}} c(\theta).$$

A special case of the cross-correlation is **auto-correlation**, where similarity is measured between $x(t)$ and $x(t-\theta)$, *i.e.*, to the signal itself,

$$a(\theta) = \langle x(t), x(t-\theta)\rangle,$$

which can be useful to say how quickly a signal is changing characteristics in some way (*e.g.*, ultrasonic signal).

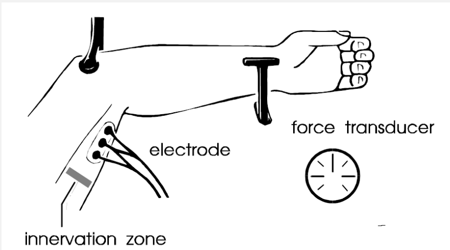
[1] signals are commonly in high dimensions, the angle between two signals are not easy to visualise, but the computation carries the same idea as how we treat 2D vectors!

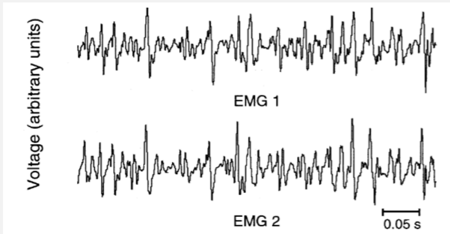
Normalized cross-correlation function, $f_c(\theta)$ The normalised cross-correlation function is defined as the normalised scalar product of the cross-correlation function,

$$f_c(\theta) = \frac{\langle x_1(t), x_2(t-\theta)\rangle}{||x_1(t)||_2 \, ||x_2(t)||_2}$$

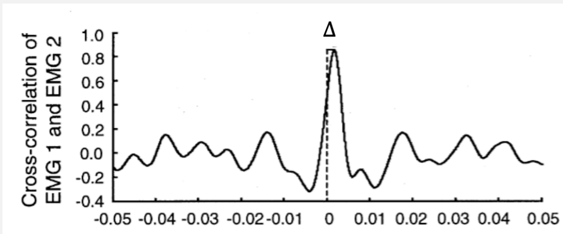
$$= \frac{\int_{-\infty}^{\infty} x_1(t)x_2(t-\theta)\mathrm{d}t}{\sqrt{\int_{-\infty}^{+\infty}|x_1(t)|^2\mathrm{d}t} \cdot \sqrt{\int_{-\infty}^{+\infty}|x_2(t-\theta)|^2\mathrm{d}t}}.$$

Example 3.1 - Measure muscle fiber conduction velocity





By applying the cross-correlation function, we are able to find the time delay between two EMG signals. By measuring the distance between two electrodes, we can calculate the conduction velocity.



3.6.3 Measuring the Difference

- An alternative way to measure the similarity between two signals is computing the **strength of their difference** (*i.e.*, norm).
- For example, using norm-2 to measure the strength of difference, we define the **mean squared error (MSE)** between two signals:

$$MSE(\theta) = ||x_1(t) - x_2(t-\theta)||_2^2$$
$$= \int_{-\infty}^{+\infty} |x_1(t) - x_2(t-\theta)|^2 \mathrm{d}t$$

  where

  - $MSE(\theta)$ is a function of the shift θ;
  - the minimum value of θ best estimates the delay;
  - MSE is the energy of the error signal.

4 Fourier Series

4.1 Orthonormal Functions

- **Orthonormal functions** has the following property:

$$\langle \phi_i(t), \phi_k(t) \rangle = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$$

  where a set of N signals $\{\phi_i(t)\}_{i=1...N}$ with this property is referred to as an **orthonormal set of signals**.

  For example, the orthogonal unitary vectors ($\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$) defining the coordinate axes (i, j, k) in a 3D Euclidean space.

- Any signals in a given space can be described by the linear combination of the basis (orthonormal) signals,

$$x(t) = \sum_{i=1}^{N} a_i \phi_i(t)$$

  where a_i are unknown complex or real numbers applied to each basis.

- The coefficients a_i can be determined by projecting the signal into each function $\{\phi_i(t)\}_{i=1...N}$:

$$\begin{aligned} \langle x(t), \phi_k(t) \rangle &= \langle \sum_{i=1}^{N} a_i \phi_i(t), \phi_k(t) \rangle \\ &= \int_{-\infty}^{\infty} \sum_{i=1}^{N} a_i \, \phi_i(t) \, \phi_k^*(t) \mathrm{d}t \\ &= \sum_{i=1}^{N} a_i \int_{-\infty}^{+\infty} \phi_i(t) \, \phi_k^*(t) \mathrm{d}t \\ &= a_k \end{aligned}$$

  - Scalar product is a linear operator.
  - The coefficients provide all the information in the signal: if we know a_k, we know the signal.
  - If the signal $x(t)$ belongs to a larger space, the projected signal will be an *approximation* of the original signal with minimum MSE.

Example 4.1 - Haar basis function

Given the signal:

$$\psi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ -1, & \frac{1}{2} < t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The following signals define a set of orthonormal basis functions:

$$\psi_{rk} = \psi(2^r t - k) \text{ for } r = 0, 1, 2... \text{ and } k = 0, 1, 2, ..., 2^r - 1$$

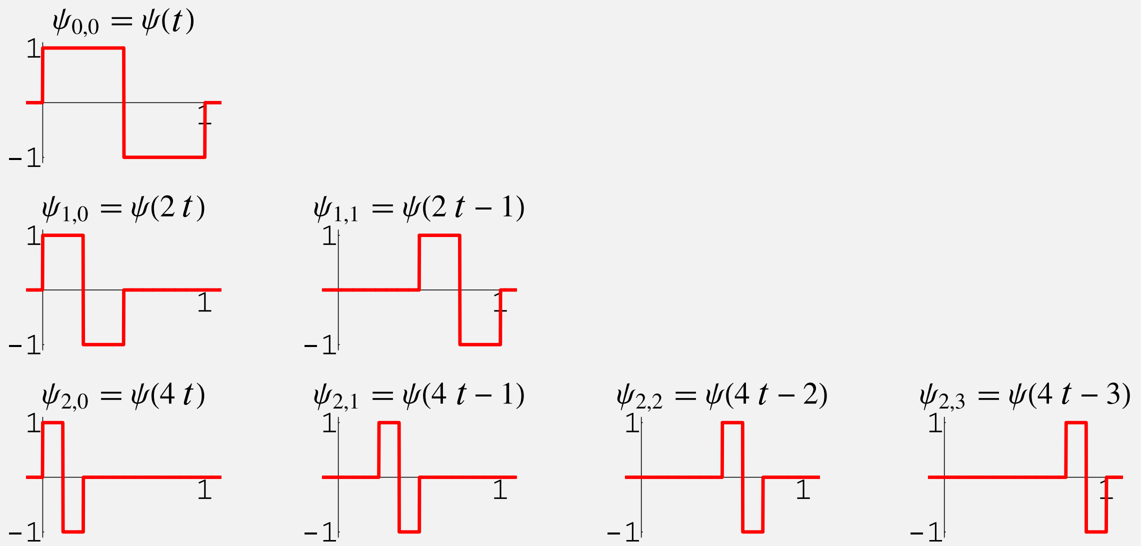


Fig. 13: *Haar basis functions*

4.2 Fourier Basis Functions

Fourier basis functions are:

$$\phi_i(t) = \frac{1}{\sqrt{T}} e^{j\omega_i t} = \frac{1}{\sqrt{T}} e^{j\frac{2\pi\cdot i}{T}t} = \underbrace{\frac{1}{\sqrt{T}}\left[\cos(\frac{2\pi\cdot i}{T}t) + j\sin(\frac{2\pi\cdot i}{T}t)\right]}_{\text{Euler's formula}},$$

where t is defined in the time interval $[0, T]$, $i \in \mathbb{Z}$ is an **integer**, and j is the imaginary unit.

These functions have the following properties,

1. periodic, the period is a function of the fundamental period T, $T_i = T/i$;
2. the frequency of each Fourier basis function is an integer multiple of fundamental (base) frequency $\frac{1}{T}$;
3. all basis functions are orthonormal to each other, when $0 \leq t \leq T$:
   - if $i \neq k$, *i.e.*, two distinguish basis functions,

$$\langle \phi_i(t), \phi_k(t) \rangle = \int_0^T \phi_i(t)\ \phi_k^*(t) \mathrm{d}t = \frac{1}{T}\int_0^T e^{j\frac{2\pi i}{T}t} e^{-j\frac{2\pi\cdot k}{T}t} \mathrm{d}t = 0,$$

   - if $i = k$,

$$\langle \phi_i(t), \phi_k(t) \rangle = \int_0^T |\phi_i(t)|^2 \mathrm{d}t = \frac{1}{T}\int_0^T \mathrm{d}t = 1$$

4.3 Fourier Series

Fourier series can be used to represent any periodic signal with finite energy in a single period.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\ e^{j\frac{2\pi\cdot k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-j\frac{2\pi\cdot k}{T}t}$$

Derivation 4.1

If we assume:

$$x(t) = \sum_{i=-\infty}^{+\infty} a_i \, \phi_i(t)$$
$$= \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{+\infty} a_k \, e^{j\frac{2\pi\cdot k}{T}t}$$

with the coefficients a_k :

$$a_k = \langle x(t), \phi_k(t) \rangle = \int_0^T x(t) \, \phi_k^*(t) \mathrm{d}t$$
$$= \frac{1}{\sqrt{T}} \int_0^T x(t) \, e^{-j\frac{2\pi\cdot k}{T}t} \mathrm{d}t$$

An equivalent expression is the **Fourier series**

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \, e^{j\frac{2\pi\cdot k}{T}t} \quad \text{with } c_k = \frac{1}{T} \int_0^T x(t) e^{-j\frac{2\pi\cdot k}{T}t} \mathrm{d}t$$

- The infinite set of orthonormal functions of the Fourier series describes any periodic signal with finite energy in a single period.

Fourier series with a finite number of terms:

$$x(t) \approx x_N(t) = \sum_{k=-N}^{+N} c_k e^{j\frac{2\pi\cdot k}{T}t}$$

The error of approximation decreases as the number of terms increases,

$$E_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} c_k \, e^{j\frac{2\pi\cdot k}{T}t}$$

$$\lim_{N\to\infty} \int_0^T |E_N(t)|^2 \mathrm{d}t = 0, \quad \text{if} \int_0^T |x(t)|^2 \mathrm{d}t < \infty$$

Fourier series converges as the number of terms increases:

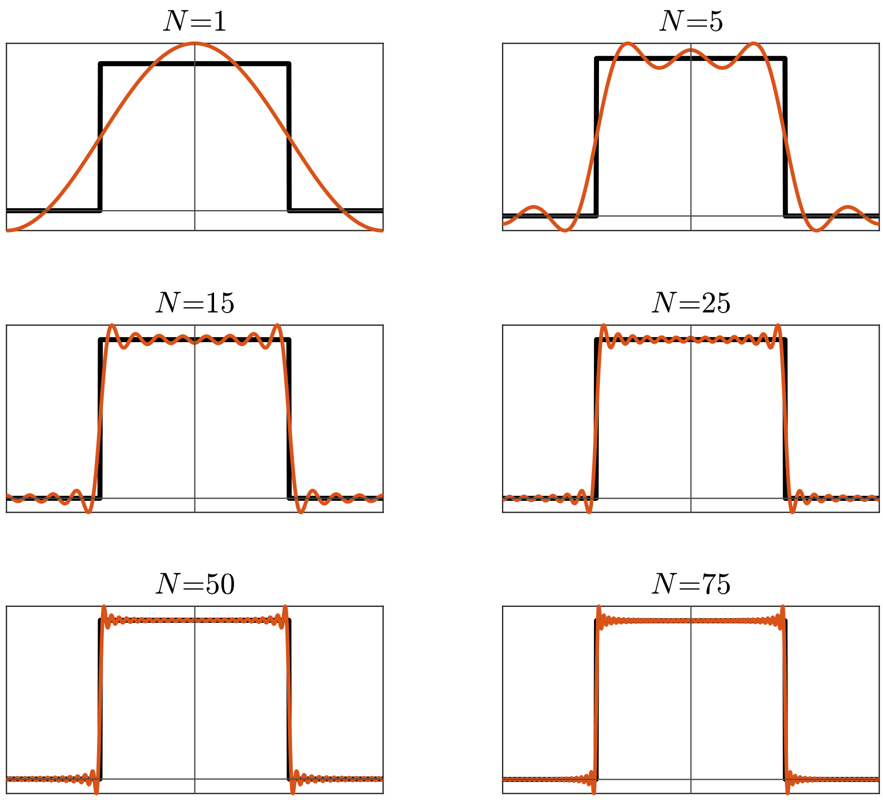


Fig. 14: *Example of convergence of the Fourier series of a periodic square wave. Number of Fourier terms (N) increased from 1 to 75.*

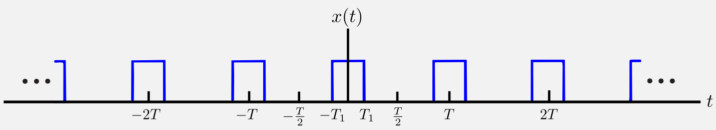
5 Fourier Transform

5.1 From Fourier Series to Fourier Transform

Example 5.1

To find a representation of any finite energy signal, not necessarily periodic: set the periodical of a period signal to *infinity*.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$

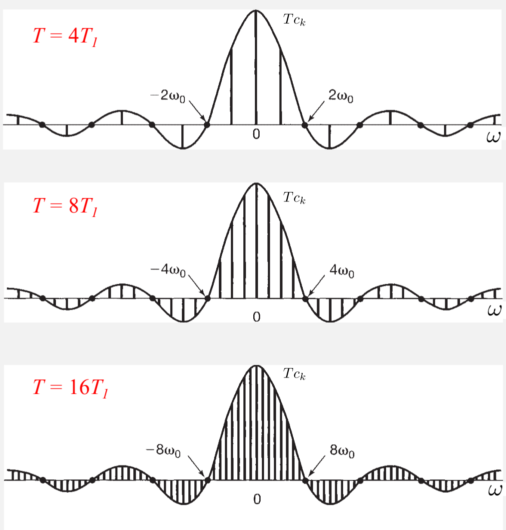


The Fourier series of the periodic signal $x(t)$ above is:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi\cdot k}{T}t}$$

with

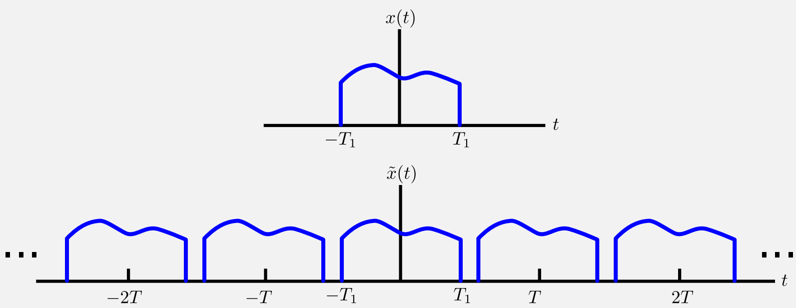
$$\begin{aligned} c_k &= \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-j\frac{2\pi\cdot k}{T}t} \\ &= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \\ &= \frac{1}{T}\,\frac{2\sin(\omega T_1)}{\omega}\bigg|_{\omega=k\omega_0} \end{aligned}$$



where $\omega_0 = \frac{2\pi}{T}$ is the frequency of the first harmonic.

Plot c_k against ω. As T increases, the frequency becomes smaller, and the points on the plot become closer.

Generalize the example above:
The signal $x(t)$ defined in the interval $t \in [-T_1, T_1]$. We build the corresponding periodic signal $\tilde{x}(t)$, which equals to $x(t)$ in one period. As $T \to \infty$, $\tilde{x}(t) \to x(t)$.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{j\frac{2\pi \cdot k}{T}t}$$

with

$$\begin{aligned} c_k &= \frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} \tilde{x}(t)e^{-j\frac{2\pi \cdot k}{T}t}\mathrm{d}t \\ &= \frac{1}{T}\underbrace{\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}\mathrm{d}t}_{X(k\omega_0)} \\ &= \frac{1}{T}X(k\omega_0) \end{aligned}$$

$x(t) \rightarrow X(\omega)$ **is known as Fourier transform.**

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} c_k \, e^{j\frac{2\pi \cdot k}{T}t} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T}X(k\omega_0)e^{j\frac{2\pi \cdot k}{T}t} \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\omega_0)\, e^{jk\omega_0 t}\, \omega_0 \end{aligned}$$

As $T \rightarrow \infty$, $\tilde{x}(t) \rightarrow x(t)$, $\omega_0 \rightarrow 0$:

$$\tilde{x}(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, \mathrm{d}\omega \tag{1}$$

$X(\omega) \rightarrow x(t)$ **is known as inverse Fourier transform.**

5.2 The Continuous-time Fourier Transform

Fourier transform:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\mathrm{d}t$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, \mathrm{d}\omega$$

- Fourier transform is a mathematical transformation employed to transform signals between the time domain and the frequency domain.
- Fourier transform is a reversible operation.
- **Fourier transform is a linear transformation**: it is defined by an integral

- For each value of ω, the Fourier transform is a complex number representing the projection of the signal on the complex exponential function $e^{j\omega t}$.

$$X(\omega) = \langle x(t), e^{j\omega t} \rangle$$

- **The Fourier transform exists for signals in the L^2 space.** These signals can be expressed as the combination of functions $e^{j\omega t}$.

Compare Fourier series and Fourier transform

Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \; e^{j\frac{2\pi \cdot k}{T} t} \quad \text{with} \quad c_k = \frac{1}{T} \int_0^T x(t) \; e^{-j\frac{2\pi k}{T} t}$$

Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \; e^{j\omega t} \; \mathrm{d}\omega$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \; e^{-j\omega t} \; \mathrm{d}t$$

The Fourier series represent *periodic* signals with *discrete* frequencies; the Fourier transform represents *non-periodic* signals with *continuous* frequencies.

5.3 Properties of Fourier Transform

5.3.1 Linearity

$$\mathcal{FT}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(\omega) + a_2 X_2(\omega)$$

Derivation 5.1

Given

$$X_1(\omega) = \int_{-\infty}^{+\infty} x_1(t) \; e^{-j\omega t} \; \mathrm{d}t \quad \text{and} \quad X_2(\omega) = \int_{-\infty}^{+\infty} x_2(t) \; e^{-j\omega t} \; \mathrm{d}t$$

$$\begin{aligned} \mathcal{FT}\{a_1 x_1(t) + a_2 x_2(t)\} &= \int_{-\infty}^{+\infty} \big(a_1 x_1(t) + a_2 x_2(t)\big) \; e^{-j\omega t} \; \mathrm{d}t \\ &= a_1 \int_{-\infty}^{+\infty} x_1(t) e^{-j\omega t} \; \mathrm{d}t + a_2 \int_{-\infty}^{+\infty} x_2(t) e^{-j\omega t} \; \mathrm{d}t \\ &= a_1 X_1(\omega) + a_2 X_2(\omega) \end{aligned}$$

5.3.2 Time shifting

$$\mathcal{FT}\{x(t - t_0)\} = e^{-j\omega t_0} \mathcal{FT}\{x(t)\} = e^{-j\omega t_0} \; X(\omega)$$

Derivation 5.2

$$\mathcal{FT}\{x(t-t_0)\} = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j\omega t}\, \mathrm{d}t$$

Let $r = t - t_0$:

$$\begin{aligned}\mathcal{FT}\{x(t-t_0)\} &= \int_{-\infty}^{+\infty} x(r)\, e^{-j\omega(r+t_0)}\, \mathrm{d}r \\ &= \int_{-\infty}^{+\infty} x(r)\, e^{-j\omega r}\, e^{-j\omega t_0} \mathrm{d}r \\ &= e^{-j\omega t_0} \int_{-\infty}^{+\infty} x(r)\, e^{-j\omega r} \mathrm{d}r\end{aligned}$$

Let $r = t$:

$$\begin{aligned}\mathcal{FT}\{x(t-t_0)\} &= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} \mathrm{d}t}_{\text{Fourier Transform}} \\ &= e^{-j\omega t_0} \mathcal{FT}\{x(t)\} \\ &= e^{-j\omega t_0}\, X(\omega)\end{aligned}$$

Question 5.1

If the Fourier transform of a signal, $f(t)$, is given by the expression of $F(\omega)$, then the Fourier transform of the signal $2f(t-3)$ is given by

(a) $2F(3\omega)$

(b) $3F(\omega/2)$

(c) $4\pi * F(\omega)$, where * denotes convolution

(d) $2e^{-3j\omega}F(\omega)$

5.3.3 Conjugation

$$\mathcal{FT}\{x^*(t)\} = X^*(-\omega)$$

if $x(t)$ is real, $X(-\omega) = X^*(\omega)$.

Derivation 5.3

The Fourier transform is:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, \mathrm{d}t$$

Take the complex conjugate:

$$X^*(\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{j\omega t}\, \mathrm{d}t$$

Change $\omega \to -\omega$:

$$\begin{aligned} X^*(-\omega) &= \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\omega t}\, \mathrm{d}t \\ &= \mathcal{FT}\{x^*(t)\} \end{aligned}$$

5.3.4 Dual property

$$\text{if } x(t) \overset{\mathcal{FT}}{\longleftrightarrow} X(\omega), \quad \text{then } X(t) \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi x(-\omega)$$

Example:

- $\delta(t) \overset{\mathcal{FT}}{\longleftrightarrow} 1, \quad 1 \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi\delta(\omega)$[2]
- $\delta(t+t_0) \overset{\mathcal{FT}}{\longleftrightarrow} e^{j\omega t_0}, \quad e^{j\omega t_0} \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$

Derivation 5.4

The Fourier transform is:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, \mathrm{d}t$$

Change $\omega \to t$, to avoid confusion, also change $t \to u$:

$$X(t) = \int_{-\infty}^{+\infty} x(u)\, e^{-jtu}\, \mathrm{d}u$$

The inverse Fourier transform of $x(-\omega)$ is:

$$\mathcal{FT}^{-1}\{x(-\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x(-\omega) e^{j\omega t} \mathrm{d}\omega$$

Change $-\omega \to u$,

$$\mathcal{FT}^{-1}\{x(u)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x(u) e^{-jut} \mathrm{d}u$$

This yields the dual property:

$$X(t) \overset{\mathcal{FT}}{\longleftrightarrow} 2\pi x(-\omega)$$

5.3.5 Time scaling

$$\mathcal{FT}\{x(at)\} = \frac{1}{|a|} X(\frac{\omega}{a})$$

a is a non-zero real number

From the above property,

$$\mathcal{FT}\{x(-t)\} = X(-\omega)$$

[2]Note that $\delta(-\omega) = \delta(\omega)$

Derivation 5.5

Fourier transform of $x(at)$ is:

$$\mathcal{FT}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)e^{-j\omega t}\mathrm{d}t$$

Replace $at \to u$, then $t = \frac{u}{a}$, $\mathrm{d}t = \frac{1}{a}\mathrm{d}u$

$$\begin{aligned}\mathcal{FT}\{x(u)\} &= \frac{1}{|a|}\int_{-\infty}^{+\infty} x(u)e^{-j\frac{\omega}{a}u}\mathrm{d}u \\ &= \frac{1}{|a|}X(\frac{\omega}{a})\end{aligned}$$

5.3.6 Parseval's relation

$$\int_{-\infty}^{+\infty} |x(t)|^2\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2\mathrm{d}\omega$$

the function $|X(\omega)|^2$ is termed the **energy-density spectrum** of the signal.

Derivation 5.6

Start from the L.H.S:

$$\begin{aligned}\int_{-\infty}^{+\infty} |x(t)|^2\mathrm{d}t &= \int_{-\infty}^{+\infty} x(t)x^*(t)\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)e^{-j\omega t}\mathrm{d}\omega\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)\left(\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\mathrm{d}t\right)\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)X(\omega)\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2\mathrm{d}\omega\end{aligned}$$

Question 5.2

The Fourier transform $X(\omega)$ of the signal $x(t)$ is

$$X(\omega) = \begin{cases} 1, & -\omega_N \le \omega \le \omega_N \\ 0, & \text{otherwise} \end{cases}$$

The energy of $x(t)$ is equal to:

(a) ω_N^2

(b) $\frac{\omega_N}{\pi}$

(c) ω_N

(d) $2\pi\omega_N$

5.3.7 Differentiation in time

$$\mathcal{FT}\left\{\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\} = j\omega X(\omega)$$
$$\mathcal{FT}\left\{\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\right\} = (j\omega)^n X(\omega)$$

5.3.8 Convolution

$$\mathcal{FT}\{x(t) * h(t)\} = X(\omega)H(\omega)$$

Derivation 5.7

$$\mathcal{FT}\{x(t) * h(t)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)e^{-j\omega t}\mathrm{d}\tau\mathrm{d}t$$

By the change of variables $t - \tau = \alpha$,

$$\begin{aligned}\mathcal{FT}\{x(t) * h(t)\} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(\alpha)e^{-j\omega(\tau+\alpha)}\mathrm{d}\tau\mathrm{d}\alpha \\ &= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}\mathrm{d}\tau \int_{-\infty}^{+\infty} h(\alpha)e^{-j\omega\alpha}\mathrm{d}\alpha \\ &= X(\omega)H(\omega)\end{aligned}$$

Convolution in the time domain is equivalent to multiplication in the Fourier domain.

5.4 More basic Fourier transforms

5.4.1 Impulse

$$x(t) = \delta(t-T) \quad \leftrightarrow \quad X(\omega) = e^{-j\omega T}$$

In particular, for $T = 0$, $X(\omega) = 1$.

5.4.2 Complex exponential

$$x(t) = e^{j\omega_0 t} \quad \leftrightarrow \quad X(\omega) = 2\pi\delta(\omega - \omega_0)$$
$$x(t) = e^{-j\omega_0 t} \quad \leftrightarrow \quad X(\omega) = 2\pi\delta(\omega + \omega_0)$$

5.4.3 Cosine

$$x(t) = A\cos(\omega_0 t) \quad \leftrightarrow \quad X(\omega) = A\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

5.4.4 Sine

$$x(t) = A\sin(\omega_0 t) \quad \leftrightarrow \quad X(\omega) = \frac{A\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Question 5.3

The Fourier transform of the signal $y(t) = x(t) \cdot \cos(\omega_0 t)$ is: (note: in all expressions below $X(\omega)$ is the Fourier transform of $x(t)$)

(a) $Y(\omega) = \frac{1}{2}[X(\omega - \omega_0) - X(\omega + \omega_0)]$

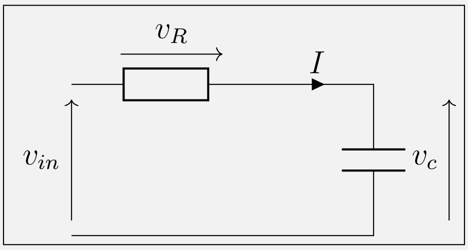
(b) $Y(\omega) = \frac{1}{2}X(\omega - \omega_0) \cdot X(\omega + \omega_0)$

(c) $Y(\omega) = \frac{1}{2}X(\omega - \omega_0)$

(d) $Y(\omega) = \frac{1}{2}[X(\omega - \omega_0) + X(\omega + \omega_0)]$

5.5 Example of application of properties of the Fourier transform

Example 5.2



For the circuit above, the input/output relation is characterized by the following differential equation:

$$v_c + RC\frac{\mathrm{d}v_c}{\mathrm{d}t} = v_{in}$$

Taking the Fourier transform (using linearity property and differentiation in time property):

$$V_c(\omega) + j\omega RCV_c(\omega) = V_{in}(\omega) \quad \rightarrow \quad V_c(\omega) = \frac{1}{1 + j\omega RC}V_{in}(\omega)$$

- The Fourier transforms are coefficients indicating the weights of each complex exponential signal $e^{-j\omega t}$ composing the signals.
- The above relation therefore tells us how the circuit processes the input signal by changing the weights of the coefficients.
- In the frequency domain, the circuit acts as a factor that multiplies each coefficient in a frequency-dependant way.

Let $H(\omega) = \frac{1}{1+j\omega RC}$, this represents the **frequency response** of the circuit.

$$V_c(\omega) = H(\omega)V_{in}(\omega)$$

Let the signal input $v_{in}(t) = e^{j\omega_0 t}$, when applying the dual property:

$$V_{in}(\omega) = 2\pi\delta(\omega - \omega_0)$$

Therefore:

$$V_c(\omega) = \frac{2\pi}{1 + j\omega RC}\delta(\omega - \omega_0) = \frac{2\pi}{1 + j\omega_0 RC}\delta(\omega - \omega_0) = H(\omega_0)2\pi\delta(\omega - \omega_0)$$

Take inverse Fourier transform:

$$v_c(t) = H(\omega_0)e^{-j\omega_0 t}$$

Since $H(\omega_0)$ is a complex number,

$$H(\omega_0) = \underbrace{|H(\omega_0)|}_{\text{magnitude}} \cdot \underbrace{e^{j\angle H(\omega_0)}}_{\text{phase}} \quad \rightarrow \quad \boxed{v_c(t) = |H(\omega_0)|e^{j(\omega_0 t + \angle H(\omega_0))}}$$

The frequency response of the circuit changes the magnitude and the phase of the complex exponential, NOT the frequency.

If the input is an impulse signal $v_{in}(t) = \delta(t)$, the response of the circuit in the frequency domain is:

$$V_c(\omega) = H(\omega)V_{in}(\omega) = H(\omega), \text{ since } V_{in}(\omega) = 1$$

And the response to the impulse in the time domain is:

$$\mathcal{FT}^{-1}\{H(\omega)\} = \frac{1}{RC}e^{\frac{t}{RC}}u(t) = \frac{1}{\tau}e^{\frac{-t}{\tau}}u(t) = h(t)$$

From the example above, there is an input-output relation in the frequency domain.

$$Y(\omega) = H(\omega)X(\omega)$$

5.5.1 Linear, Time-Invariant (LTI) Systems

Definition of LTI systems

Define a system operator $T\{\cdot\}$ where a mapping relation between the system input $x(t)$ and output $y(t)$ exists: $y(t) = T\{x(t)\}$.

Linear If the system is linear and the input is scaled by a constant A, then the output will be scaled by the same constant A.

$$Ay(t) = T\{Ax(t)\}$$

Time-invariant If the system is time-invariant and we delay the input by τ, then the output will also be delayed by the same amount τ.

$$y(t-\tau) = T\{x(t-\tau)\}$$

Therefore, an LTI system also holds the property that

$$A_0 y(t) + A_1 y(t-\tau) = T\{A_0 x(t) + A_1 x(t-\tau)\}$$

For any continuous-time signals:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau$$

For any signal $x(t)$:

$$y(t) = T\{x(t)\} = T\left\{\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau\right\} = \int_{-\infty}^{+\infty} x(\tau)T\left\{\delta(t-\tau)\right\}\mathrm{d}\tau$$

If $h(t) = T\{\delta(\tau)\}$ (response of the system to the **impulse response**):

$$T\{\delta(t-\tau)\} = h(t-\tau), \text{then} : y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\mathrm{d}\tau = x(t) * h(t)$$

The output of an LTI system is the convolution of the input with the impulse response, *i.e.*, the system is fully defined by the impulse response.

Therefore, in the frequency domain, the relation $Y(\omega) = H(\omega)X(\omega)$ holds for all LTI systems. (Why? Convolution in the time domain is equivalent to the multiplication in the Fourier domain.)

5.6 Magnitude and Phase Spectra

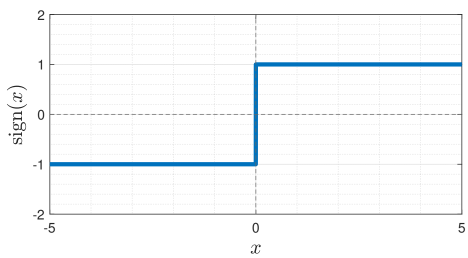
In general, $X(\omega)$ is a **complex** function of ω:

$$X(\omega) = a(\omega) + \mathbf{j}b(\omega) = |X(\omega)|e^{j\angle X(\omega)}$$

- $|X(\omega)|$ is **magnitude**, it describes the basic frequency content of a signal, *i.e.* the relative magnitudes of the complex exponentials that make up $x(t)$.
- $\angle X(\omega)$ is **phase angle**, it determines the different look of signals, even if the magnitude remains unchanged.

$$\angle X(\omega) = \tan^{-1}\left[\frac{\Im\{X(\omega)\}}{\Re\{X(\omega)\}}\right] + \pi\left[\frac{1-\text{sign}(\Re\{X(\omega)\})}{2}\right]$$

$$\text{where } \text{sign}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



Example 5.3

A ship encounters the superposition of three wave trains, each of which can be modelled as a sinusoidal signal.

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3)$$

With fixed magnitudes for these sinusoids, **the amplitude of their sum may be quite small or very large, depending on the relative phases.** The implications of phase for the ship, therefore, are quite significant.

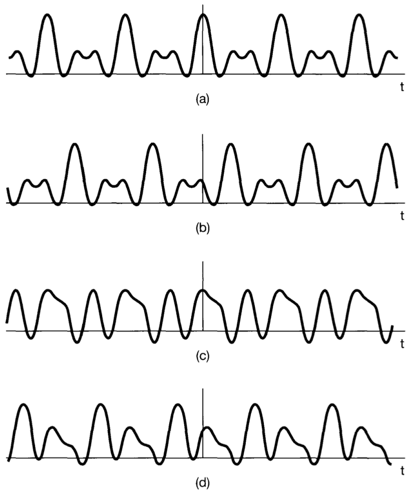


Figure 6.1 The signal $x(t)$ given in eq. (6.3) for several different choices of the phase angles ϕ_1, ϕ_2, and ϕ_3: (a) $\phi_1 = \phi_2 = \phi_3 = 0$; (b) $\phi_1 = 4$ rad, $\phi_2 = 8$ rad, $\phi_3 = 12$ rad; (c) $\phi_1 = 6$ rad, $\phi_2 = -2.7$ rad, $\phi_3 = 0.93$ rad; (d) $\phi_1 = 1.2$ rad, $\phi_2 = 4.1$ rad, $\phi_3 = -7.02$ rad.

(Example adopted from **Signals and Systems, 2nd Edition**, P424)

Specifically, for the circuit example above:

$$|H(\omega_0)| = \frac{1}{\sqrt{1+\omega^2(RC)^2}} = \frac{1}{\sqrt{1+(\frac{\omega}{\omega_c})^2}}$$

$$\angle H(\omega_0) = -\tan^{-1}(\omega RC) = -\tan^{-1}(\frac{\omega}{\omega_c})$$

5.7 Fourier Transform of Periodic Signals

If $x(t)$ is a periodic signal:

$$\mathcal{FT}\{x(t)\} = 2\pi \sum_{k=-\infty}^{+\infty} c_k \delta(\omega - k\omega_0)$$

where

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-j\frac{2\pi k}{T}t}\mathrm{d}t$$

Derivation 5.8

Fourier series of a periodic signal $x(t)$ with period T is:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi \cdot k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)e^{-j\frac{2\pi \cdot k}{T}t}\mathrm{d}t$$

By applying linearity property:

$$\begin{aligned} X(\omega) &= \mathcal{FT}\Big\{ \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi\cdot k}{T}t} \Big\} \\ &= \sum_{k=-\infty}^{+\infty} c_k \, \mathcal{FT}\{e^{jk\omega_0 t}\} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} c_k \delta(\omega - k\omega_0) \end{aligned}$$

5.7.1 Fourier transform of a train of impulses

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \xrightarrow{\mathcal{FT}} X(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

where

$$\omega_0 = \frac{2\pi}{T}$$

Derivation 5.9

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - T)$$

From the definition above, $x(t)$ is periodic with period T:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi\cdot k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta(t) e^{-j\frac{2\pi\cdot k}{T}t} \mathrm{d}t = \frac{1}{T}$$

Thus:

$$X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_0) \quad \text{with } \omega_0 = \frac{2\pi}{T}$$

Question 5.4

The Fourier transform of the time-domain signal $x(t) = \left[e^{-t}u(t)\right] \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT)$ is:

(note: $u(t)$ is the unit step function)

(a) $X(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} \frac{1}{1+j(\omega - n\frac{2\pi}{T})}$

(b) $X(\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{-j(\omega - n\frac{2\pi}{T})}$

(c) $X(\omega) = \frac{1}{1+j\omega} \cdot \frac{1}{T}\sum_{n=-\infty}^{+\infty} \delta(\omega - n\frac{2\pi}{T})$

(d) $X(\omega) = e^{-j\omega} \cdot \frac{1}{T}\sum_{n=-\infty}^{+\infty} \delta(\omega - n\frac{2\pi}{T})$

6 Sampling Theorem

To obtain a discrete-time signal from a continuous-time signal, we need a **C/D converter**.

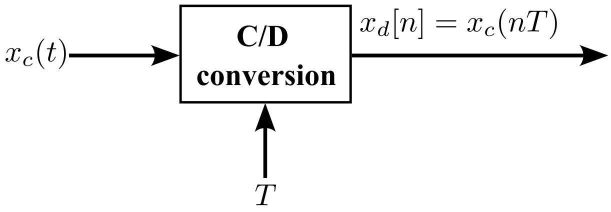


Fig. 15: *A C/D converter*

Mathematically,

$$x[n] = x_c(nT) \quad -\infty < n < +\infty$$

where T is sampling period, $f_s = \frac{1}{T}$ is sampling frequency.

- In general, the C/D transformation cannot be inverted.
- Infinite continuous signals can reproduce a given sequence of samples,

An ideal C/D converter applies the T property so that the sampling can be done without losing information.

6.1 Sampling Process

Impulse train modulator $s(t)$ is:

$$s(t) = \sum_{-\infty}^{+\infty} \delta(t - nT)$$

The sampled signal $x_s(t)$ is obtained by multiplying the impulse train modulator (Figure 16.b) with the continuous-time signal $x_c(t)$ of interest (Figure 16.a):

$$\begin{aligned} x_s(t) &= x_c(t)\ s(t) \\ &= \sum_{n=-\infty}^{+\infty} x_c(t)\delta(t - nT) \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT) \end{aligned}$$

The sampled signal, $x_s(t)$, is still defined in the continuous-time domain, but it contains all information in the sampled discrete-time domain.

Apply the Fourier transform to $x_s(t)$:

$$\begin{aligned} X_s(\omega) &= \mathcal{FT}\{x_c(t)\} \cdot \mathcal{FT}\{s(t)\} \\ &= \frac{1}{2\pi} X_c(\omega) * \mathcal{FT}\{s(t)\} \\ &= \frac{1}{T} X_c(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - k\omega_s) \\ &= \boxed{\frac{1}{T} \sum_{n=-\infty}^{+\infty} X_c(\omega - k\omega_s)} \end{aligned}$$

where sampling frequency $\omega_s = \omega_0 = \frac{2\pi}{T}$.

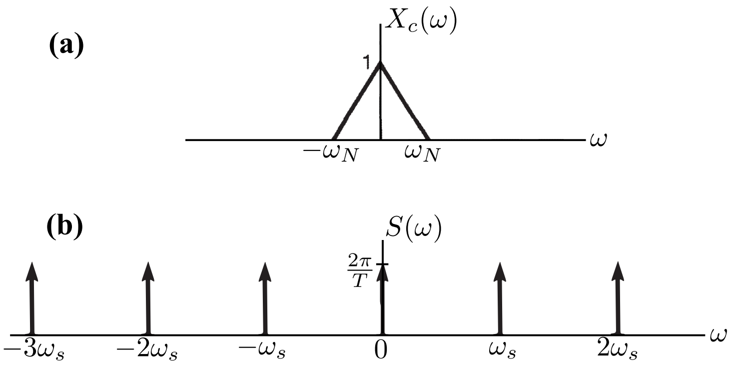


Fig. 16: *Spectrum of (a) the original continuous-time signal pending to be sampled; (b) sampling signal (delta train)*

For sampled signals: ω_N is the signal bandwidth

- if $\omega_s \geq 2\omega_N$, the replicas in the periodization do not overlap (Figure 17.a)
- if $\omega_s < 2\omega_N$, the replicas overlap, also known as **aliasing**[3] (Figure 17.b).

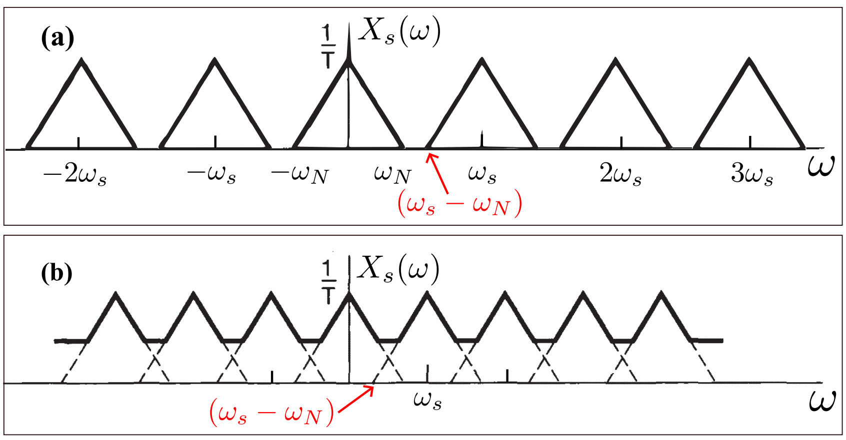


Fig. 17: *(a) Sampling without aliasing, $\omega_s \geq 2\omega_N$; (b) Sampling with aliasing, $\omega_s < 2\omega_N$*

6.1.1 Nyquist-Shannon Sampling Theorem

Nyquist-Shannon sampling theorem states that: **to retain the ability to reproduce (reconstruct) the original signal, the minimum sampling frequency during signal sampling must be at least twice its frequency.**

Mathematically, let $x_c(t)$ be a band-limited signal with $X_c(\omega) = 0$, for $|\omega| \geq \omega_N$. Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, if

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_N$$

where $2\omega_N$ is the minimal sampling rate and referred to as the **Nyquist rate**.

[3]"alias" is a Latin word, meaning "otherwise", or "elsewhere".

Nyquist-Shannon sampling theorem provides the condition under which the C/D transformation **can be inverted** without losing information, as shown in Figure 18.

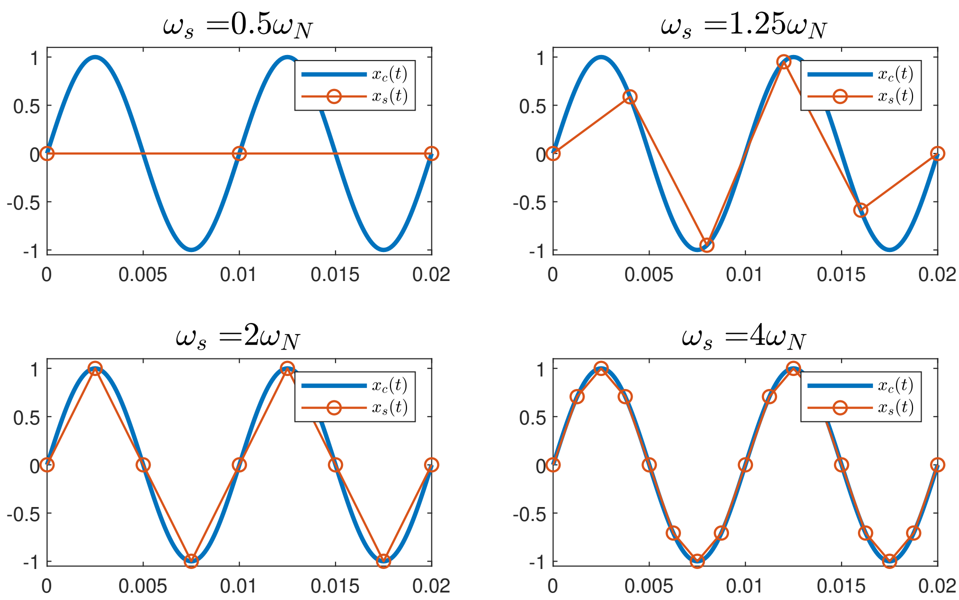


Fig. 18: *Sampling of a continuous signal* $x_c(t) = \sin(200\pi t)$*: (left to right, top to bottom)* $\omega_s = 0.5\omega_N$, $\omega_s = 1.25\omega_N$, $\omega_s = 2\omega_N$, $\omega_s = 4\omega_N$. *According to the Nyquist-Shannon sampling theorem, aliasing occurs when* $\omega_s < 2\omega_N$

Question 6.1

Consider the signal $x_1(t) = x(t) \cdot \cos(\omega_0 t)$ with $\omega_0 \neq 0$. The signal $x(t)$ has bandwidth $\omega_N \leq \omega_0$. Which of the minimum sampling frequency $f_{s,min}$ to sample $x_1(t)$ without loss of information?

(a) $f_{s,min} = 2(\omega_0 + \omega_N)$

(b) $f_{s,min} = 2(\omega_0 - \omega_N)$

(c) $f_{s,min} = 2\omega_0$

(d) $f_{s,min} = 2\omega_0\omega_N$

6.2 Reconstruction Process

Ideal low-pass filters can be used to reconstruct the signals.

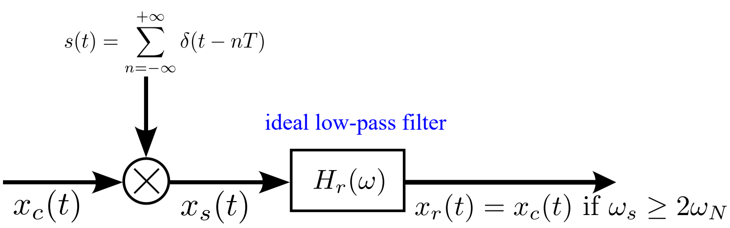


Fig. 19: *A low-pass filter system for signal reconstruction*

1. Apply an ideal low-pass filter $H_r(\omega)$ to the sampled signal, $X_s(\omega)$.

$$X_r(\omega) = X_s(\omega) \cdot H_r(\omega)$$

This removes the redundant **replicated** sampled signal in the frequency domain, *i.e.*, we only keep one signal. This process is shown in Figure 21.

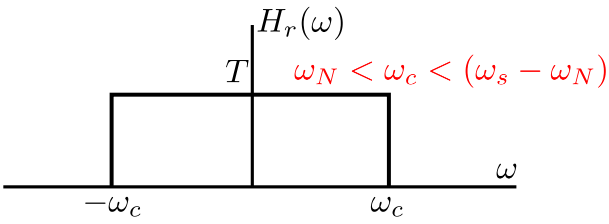


Fig. 20: *An ideal low-pass filter* $H_r(\omega)$

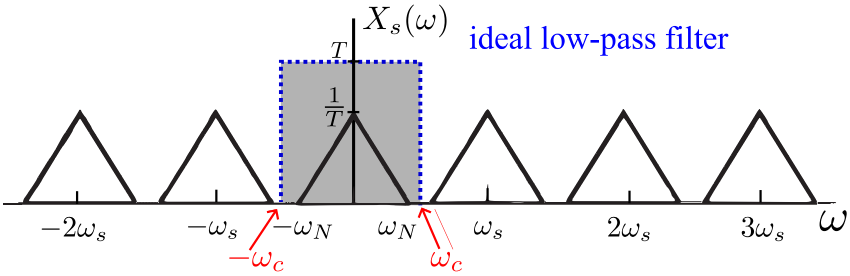


Fig. 21: *Truncate the sample signal* $X_s(\omega)$ *with an ideal low-pass filter* $H_r(\omega)$

2. Due to the convolution property:

$$X_c(\omega) = \mathcal{FT}\{x_s(t) * h_r(t)\}$$

3. Apply inverse Fourier transform:

$$\begin{aligned} x_c(t) &= x_s(t) * h_r(t) \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT) * h_r(t) \\ &= \boxed{\sum_{n=-\infty}^{+\infty} x_c(nT) h_r(t - nT)} \\ &= \sum_{n=-\infty}^{+\infty} x_c(nT) \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T} \end{aligned}$$

with

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

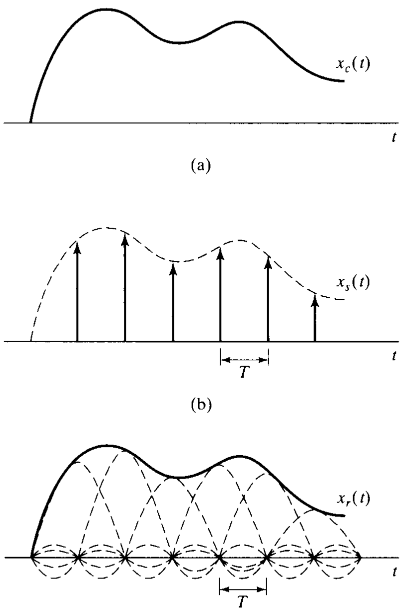


Fig. 22: *Reconstruction*