Pre-sessional learning for iBSc students
Section 7: Verifying solutions of partial differential equations

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The diffusion equation in 1D is given by

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2},$$

where C denotes a physical quantity to be conserved *e.g.*, temperature distribution in heat transport, concentration in mass transport, or velocity in conservation of momentum. C has both temporal and spatial dependency, *i.e.*, $C(x,t)$. D is the diffusivity, which quantifies how fast the quantity (heat, mass, momentum) diffuses.

Question 1: Verify that $C(x,t) = C_0$ satisfies the 1D heat equation, where C_0 is a constant.

Since C_0 is a constant, the derivatives w.r.t. t and x are both 0,

$$\frac{\partial C}{\partial t} = 0, \quad \frac{\partial^2 C}{\partial x^2} = 0.$$

This equates L.H.S. and R.H.S. of the 1D diffusion equation. ✓

Question 2: Verify that $C(x,t) = C_0(1 + \frac{x}{L})$ also satisfies the 1D heat equation, where C_0 and L are constants.

1. Differentiate $C(x,t)$ w.r.t. t yields 0.

2. Differentiate $C(x,t)$ w.r.t. x yields

$$\frac{\partial C(x,t)}{\partial x} = \frac{C_0}{L}, \quad \frac{\partial^2 C(x,t)}{\partial x^2} = 0.$$

This equates L.H.S. and R.H.S. of the 1D diffusion equation. ✓

Question 3: **Verify that $C(x,t) = \sqrt{\frac{1}{4\pi Dt}}\, e^{-x^2/4Dt}$ also satisfies the 1D heat equation.**

1. Differentiate $C(x,t)$ w.r.t. t yields

$$
\begin{aligned}
\frac{\partial C(x,t)}{\partial t} &= \sqrt{\frac{1}{4\pi D}} \cdot \left(-\frac{1}{2}t^{-\frac{3}{2}}\right) \cdot e^{-x^2/4Dt} + \sqrt{\frac{1}{4\pi Dt}} \cdot \left(\frac{x^2}{4D}t^{-2}\right) \cdot e^{-x^2/4Dt} \\
&= \left(-\frac{1}{2}t^{-1}\right) \underbrace{\sqrt{\frac{1}{4\pi Dt}}\, e^{-x^2/4Dt}}_{C(x,t)} + \left(\frac{x^2}{4D}t^{-2}\right) \underbrace{\sqrt{\frac{1}{4\pi Dt}}\, e^{-x^2/4Dt}}_{C(x,t)} \\
&= C(x,t) \cdot \left(-\frac{1}{2}t^{-1} + \frac{x^2}{4D}t^{-2}\right).
\end{aligned}
$$

2. Differentiate $C(x,t)$ w.r.t. t yields

$$
\begin{aligned}
\frac{\partial C(x,t)}{\partial x} &= C(x,t) \cdot \left(-\frac{x}{2Dt}\right), \\
\frac{\partial^2 C(x,t)}{\partial x^2} &= C(x,t) \cdot \left(-\frac{x}{2Dt}\right)^2 + C(x,t) \cdot \left(-\frac{1}{2Dt}\right) \\
&= C(x,t) \left[\left(-\frac{x}{2Dt}\right)^2 + \left(-\frac{1}{2Dt}\right)\right].
\end{aligned}
$$

3. Multiply $\frac{\partial^2 C(x,t)}{\partial x^2}$ by D (which is the R.H.S. of the 1D diffusion equation)

$$
\text{(R.H.S.)}\;\; D \cdot C(x,t) \left[\left(-\frac{x}{2Dt}\right)^2 + \left(-\frac{1}{2Dt}\right)\right] = C(x,t) \left(\frac{x^2}{2Dt^2} - \frac{1}{2t}\right) \;\; \text{(L.H.S.)} \quad \checkmark
$$

The Schrödinger equation in 1D is given by

$$
i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2},
$$

where ψ is the wave function that describes the quantum state of a partial, and i is the imaginary unit (*i.e.*, $i^2 = -1$).

Question 4: **Verify that $\psi(x,t) = e^{i(kx-\omega t)}$ is a solution of the 1D Schrödinger equation if $\omega = k^2$.**

1. Differentiate $\psi(x,t)$ w.r.t. t yields

$$
\frac{\partial \psi(x,t)}{\partial t} = -ik^2 \cdot e^{i(kx-k^2 t)}.
$$

2. Differentiate $\psi(x,t)$ w.r.t. x yields

$$
\begin{aligned}
\frac{\partial \psi(x,t)}{\partial x} &= ik \cdot e^{i(kx-k^2 t)}, \\
\frac{\partial^2 \psi(x,t)}{\partial x^2} &= (ik) \cdot (ik) \cdot e^{i(kx-k^2 t)} = -k^2 e^{i(kx-k^2 t)}.
\end{aligned}
$$

3. Multiply $\frac{\partial \psi(x,t)}{\partial t}$ by i (which is the L.H.S. of the 1D Schrödinger equation)

$$\text{(L.H.S.)}\ \ i \cdot -ik^2 \cdot e^{i(kx-k^2t)} = k^2 \cdot e^{i(kx-k^2t)} = -1 \cdot -k^2 e^{i(kx-k^2t)}\ \ \text{(R.H.S.)} \quad ✓$$

Question 5: Verify that $\psi(x,t) = e^{-(kx+i\omega t)}$ is also a solution of the 1D Schrödinger equation if $\omega = -k^2$.

1. Differentiate $\psi(x,t)$ w.r.t. t yields

$$\frac{\partial \psi(x,t)}{\partial t} = -ik^2 \cdot e^{-(kx+ik^2t)}.$$

2. Differentiate $\psi(x,t)$ w.r.t. x yields

$$\frac{\partial \psi(x,t)}{\partial x} = -k \cdot e^{-(kx+ik^2t)},$$
$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = k^2 \cdot e^{-(kx+ik^2t)}.$$

3. Multiply $\frac{\partial \psi(x,t)}{\partial t}$ by i (which is the L.H.S. of the 1D Schrödinger equation)

$$\text{(L.H.S.)}\ \ i \cdot -ik^2 \cdot e^{-(kx+ik^2t)} = k^2 \cdot e^{-(kx+ik^2t)} = -1 \cdot k^2 \cdot e^{-(kx+ik^2t)}\ \ \text{(R.H.S.)} \quad ✓$$